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The Diffusion Phenomenon for Dissipative Wave Equations with Time-Dependent Operators

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To the Graduate Council:

I am submitting herewith a dissertation written by Montgomery Taylor entitled "The Diffusion Phenomenon for Dissipative Wave Equations with Time-Dependent Operators." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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The Diffusion Phenomenon for Dissipative Wave Equations in Metric Measure Spaces with Time-Dependent Operators

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To my amazing wife Kat, who has been an incredible source of support and encouragement.

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Abstract

In this dissertation, we study the long-time behavior of the solution to a type of dissipative wave equation, where the operator in this equation is time-dependent and self-adjoint, and the solution to this equation is defined in a metric measure space satisfying appropriate conditions.

We first consider the solution to an important particular dissipative wave equation in Euclidean space, where the operator is uniformly elliptic and in divergence form for each fixed time. We derive the asymptotic behavior of the solution to this equation. Furthermore, the work done for this particular problem serves as a stepping stone, allowing us to study the solution to the general type of dissipative wave equation.

When we study the general problem, the operator in the dissipative wave is assumed to correspond to a Dirichlet form. We link hyperbolic PDEs with the firmly established theories for parabolic PDEs and Dirichlet forms, subsequently deriving the asymptotic behavior of the solution to the general dissipative wave equation.

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Chapter 1

Introduction

1.1 Basic definitions and assumptions

Throughout this dissertation, we assume that X is a metric space with a complete metric d. Let τ be the topology on X induced by d, and assume that (X, τ) is separable. Also, let m be a positive Radon measure with respect to (X, τ). We will refer to the triple (X, d, m) as a metric measure space.

For $1 \leq p < \infty$, we define the function space $L^p(X)$ to be the collection of all $f : X \to \mathbb{R}$ such that $\int_X |f(x)|^p dm(x) < \infty$; note that the definition of the above spaces depends on the measure m. We define the norm $\|\cdot\|_{L^p(X)}$ on $L^p(X)$ via $\|f\|_{L^p(X)} := \left(\int_X |f(x)|^p dm(x)\right)^{1/p}$. The space $\mathbb{H} := \left(L^2(X), \langle \cdot, \cdot \rangle_{L^2(X)}\right)$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle_{L^2(X)}$ defined via $\langle f, g \rangle_{L^2(X)} := \int_X f(x)g(x)dm(x)$. We also have the identity $\|f\|^2_{L^2(X)} = \langle f, f \rangle_{L^2(X)}$.

Let $L : \mathbb{H} \to \mathbb{H}$ be an operator with domain $D(L)$ that is dense in $\mathbb{H}$; in this case, we say that L is densely defined. Any densely defined operator L admits a unique adjoint operator L^* such that $\langle Lf, g \rangle_{L^2(X)} = \langle f, L^* g \rangle_{L^2(X)}$ for $f \in D(L)$ and $g \in D(L^*)$; the operator L is self-adjoint if L and L^* are equal as operators on $\mathbb{H}$. An operator L is positive in $\mathbb{H}$ if $\langle Lf, f \rangle_{L^2(X)} \geq 0$ for all $f \in \mathbb{H}$.

Let $\mathcal{Q} : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ be a symmetric bilinear form, meaning that the domain of $\mathcal{Q}$ is $D(\mathcal{Q}) \times D(\mathcal{Q})$ and $\mathcal{Q}(f, g) = \mathcal{Q}(g, f)$ for $f, g \in D(\mathcal{Q})$. Note that $D(\mathcal{Q})$ represents the component-wise domain of $\mathcal{Q}$, not its full domain. We assume that $D(\mathcal{Q})$ is dense in $\mathbb{H}$, and $\mathcal{Q}(f, f) \geq 0$ for $f \in D(\mathcal{Q})$. We say $\mathcal{Q}$ is closed if $D(\mathcal{Q})$ is a Hilbert space when endowed with the inner product $\mathcal{Q}(f,g) + \langle f,g\rangle_{L^2(X)}$. The form $\mathcal{Q}$ is a Dirichlet form if it is closed and satisfies: 1) $g = \min\{\max\{f,0\},1\} \in D(\mathcal{Q})$ if $f \in D(\mathcal{Q})$ and 2) $\mathcal{Q}(g,g) \leq \mathcal{Q}(f,f)$.

Example 1.1. *The prototypical Dirichlet form on* $\mathbb{R}^N$ *is* $\mathcal{Q}(f,g) = \int_{\mathbb{R}^N} a(x)\nabla f(x)\cdot\nabla g(x)\,dx$*, where for instance* $0 \leq a(x) \in C^1(\mathbb{R}^N)$*. This form corresponds to the self-adjoint operator* $Lf = -\nabla\cdot(a(x)\nabla f(x))$*. When reading this dissertation, keep this example in mind when Dirichlet forms are referred to.*

In general, Dirichlet forms are definable within the context of many types of metric measure spaces with no relation to partial derivatives. That is to say, in general, $\mathcal{Q}(f,g) = \int_X d\mathbb{Q}(f,g)$, where $\mathbb{Q}(f,g)$ is a signed Radon measure on X. We say that $\mathbb{Q}$ is the energy measure form associated with $\mathcal{Q}$.

A Dirichlet form $\mathcal{Q}$ is strongly local if $\mathcal{Q}(f,g) = 0$ whenever f is constant on a neighborhood of the support of g. Also a Dirichlet form $\mathcal{Q}$ is regular if $D(\mathcal{Q}) \cap C_c(X)$ is: 1) dense in $D(\mathcal{Q})$ with respect to the norm $\left(\mathcal{Q}(f,f) + \|f\|^2_{L^2(X)}\right)^{1/2}$ and 2) dense in $C_c(X)$ with the uniform norm.

We can extend $\mathcal{Q}$ and $\mathbb{Q}$ in the following way. An $m-$measurable function f is "locally" in $D(\mathcal{Q})$ if for every relatively compact open set $G \subset X$, there exists $f_G \in D(\mathcal{Q})$ such that $f = f_G$ $m-$a.e. on G. The set $D(\mathcal{Q})_{loc}$ is the collection of all such functions f. Consequently, $\mathbb{Q}(f,f)$ is a Radon measure on X for each $f \in D(\mathcal{Q})_{loc}$.

1.2 Description of the problem

Let (X,d,m) be a separable metric measure space. For each $t \in \mathbb{R}$, let $A(t) : \mathbb{H} \to \mathbb{H}$ be a densely defined, self-adjoint and positive operator in $\mathbb{H}$. We seek to describe the long-time behavior of the solution to

$$\begin{cases} u_{tt}(x,t) + u_t(x,t) + A(t)u(x,t) = 0, & x \in X,\ t > 0, \\ (u,u_t)(x,0) = (u_0,u_1)(x), & x \in X, \end{cases} \tag{1.1}$$

where $u_t := \partial_t u$ and $u_{tt} := \partial_t^2 u$. Under appropriate conditions, we are able to show that the asymptotic behavior of the solution to (1.1) aligns with the asymptotic behavior of a prescribed solution of

$$v_t(x,t) + A(t)v(x,t) = 0, \quad x \in X,\ t > 0, \tag{1.2}$$

meaning the solution of (1.1) exhibits the diffusion phenomenon; note that $v_t := \partial_t v$. More precisely, $u(x,t)$ exhibits the diffusion phenomenon if

$$\|u(x,t) - v(x,t)\|_{L^2(X)} \leq C\ (t+1)^{-K}\ \|(u_0,u_1)\| \tag{1.3}$$

for some $K > 0$, where $\|(u_0,u_1)\|$ represents the appropriate norm for the data u_0 and u_1. We will clarify what the appropriate norm for the data is later. The constant K and the rate of decay rate for $\|v(x,t)\|_{L^2(X)}$ depend on properties of the metric measure space (X,d,m) and the operator $A(t)$. In principle, we expect that $\|v(x,t)\|_{L^2(X)}\,(t+1)^K \to \infty$ as $t \to \infty$. Hence, as a consequence of (1.3), we expect that $\|u(x,t)\|_{L^2(X)}$ behaves like $\|v(x,t)\|_{L^2(X)}$ as $t \to \infty$.

One of our greatest challenges is determining how the behavior of the solution u to (1.1) relates to properties of the operator $A(t)$. Hence, we do not focus on $A(t)$ "directly." Instead, we focus our attention on the bilinear form $\mathcal{E}_t : D(\sqrt{A(t)}) \times D(\sqrt{A(t)}) \to \mathbb{R}$ defined via $\mathcal{E}_t(f,g) := \langle\sqrt{A(t)}f, \sqrt{A(t)}g\rangle_{L^2(X)}$. In many cases, $\langle\sqrt{A(t)}f, \sqrt{A(t)}g\rangle_{L^2(X)}$ can be rewritten in a convenient way, allowing us determine the behavior of u under relatively mild assumptions. For instance, in chapter 2, we have $X = \mathbb{R}^N$ and $A(t)$ defined via $A(t)f(x) := -\nabla \cdot (a(x,t)\nabla f(x))$ for $f \in D(A(t))$. In this case, $\mathcal{E}_t$ is defined by $\mathcal{E}_t(g,h) := \int_{\mathbb{R}^N} a(x,t)\nabla g(x) \cdot \nabla h(x)dx$ for $g,h \in D(\sqrt{A(t)})$. Consequently, the behavior of u depends on the coefficient $a(x,t) \geq 0$. We discuss $\mathcal{E}_t$ further in section 1.3, and we introduce some of the machinery.

1.3 Dirichlet forms and the intrinsic metric

Remark 1.2. *The intrinsic metric ρ, defined in (1.4), determines the behavior of the solution u to (1.1). However, in many cases, ρ is essentially "invisible." For example, see chapter 2, where $X = \mathbb{R}^N$ and $A(t)$ in (1.1) is defined via $A(t)f(x) := -\nabla \cdot (a(x,t)\nabla f(x))$. Here,*

$a_1 \leq a(x,t) \leq a_2$ for constants $a_1, a_2 > 0$, and this results in ρ being the Euclidean metric on $\mathbb{R}^N$. Consequently, chapter 2 and thus the main procedures for showing the diffusion phenomenon (1.3) can be understood without knowledge of the machinery presented in this section.

We reframe problem (1.1) in the sense that we do not start with an operator $A(t)$. Instead, for each $t \in \mathbb{R}$, we start with a strongly local, regular Dirichlet form $\mathcal{E}_t$ having domain $D(\mathcal{E}_t)$, and then we obtain $A(t)$ via Fukushima, Oshima and Takeda [6, Theorem 1.3.1]. This theorem states that given a Dirichlet form $\mathcal{E}_t$, there exists a densely defined, self-adjoint and positive operator $A(t)$ such that $\mathcal{E}_t(f,g) = \langle \sqrt{A(t)}f, \sqrt{A(t)}g \rangle_{L^2(X)}$ for $f, g \in D(\mathcal{E}_t)$.

The first significant assumption we make is that there exists a strongly local, regular Dirichlet form $\mathcal{E}$ on $L^2(X)$ that serves as the reference form, meaning

$$c_1\mathcal{E}(f,f) \leq \mathcal{E}_t(f,f) \leq c_2\mathcal{E}(f,f) \tag{D}$$

for all $f \in D(\mathcal{E})$ and $t \in \mathbb{R}$, where the constants c_1 and $c_2 > 0$. We also assume that $D(\mathcal{E}_t) = D(\mathcal{E})$ for all $t \in \mathbb{R}$.

Let Γ be the energy measure form associated with the reference Dirichlet form $\mathcal{E}$. The form $\mathcal{E}$ is closely linked to the behavior of u via the intrinsic pseudo metric ρ on X, which is defined by

$$\rho(x,y) := sup\{f(x) - f(y) : f \in D(\mathcal{E})_{loc} \cap C(X), d\Gamma(f,f) \leq dm(x) \text{ on } X\}, \tag{1.4}$$

where "$\leq$" here means less than or equal to as measures on X. We now "forget" about the original metric d, replacing it with ρ. For $x \in X$, define the $\rho-$ball $B_R^\rho(x) := \{z \in X : \rho(z,x) < R\}$. It will be shown later that the behavior of $m(B^\rho_{\sqrt{t}}(x))$ and $exp\left(-\frac{\rho(x,z)^2}{t}\right)$ as $t \to \infty$ essentially dictate the decay rate for $\|u(x,t)\|_{L^2(X)}$ as $t \to \infty$.

In the case when $X = \mathbb{R}^N$, d is the Euclidean metric, m is the Lebesgue measure and $d\Gamma(f,f) = a(x)|\nabla f(x)|^2\, dx$, then $\rho = d$, if $a(x) \equiv 1$. If $a(x) < 1$ in some open set U, then ρ may be viewed as a version of d that has been "stretched" in U, i.e., $d(y,z) \leq \rho(y,z)$ for

$y, z \in U$. Similarly, if $a(x) > 1$ in U, then points in U have been "compressed" together, i.e., $\rho(y, z) \leq d(y, z)$.

1.4 History

Matsumura [22] considered the problem

$$\begin{cases} u_{tt}(x,t) + u_t(x,t) - \Delta u(x,t) = 0, & x \in \mathbb{R}^N,\ t > 0, \\ (u, u_t)(x, 0) = (u_0, u_1)(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.5}$$

and he used Fourier methods to establish sharp $L^p(\mathbb{R}^N) - L^q(\mathbb{R}^N)$ decay estimates for the solution to (1.5) and its space and time derivatives. In particular, for $u_0, u_1 \in C_c^\infty(\mathbb{R}^N)$, he showed

$$\left\| \partial_t^i \partial_x^\alpha u(x,t) \right\|_{L^2(\mathbb{R}^N)} \leq C\,(t+1)^{-N/4 - i - |\alpha|/2} \left\| (u_0, u_1) \right\|, \tag{1.6}$$

where $\alpha := (\alpha_1, \ldots, \alpha_N)$, $|\alpha| := \alpha_1 + \cdots + \alpha_N$ and $\partial_x^\alpha := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_N}^{\alpha_N}$.

Inequality (1.6) offers us a key insight as to why we expect solutions to damped wave equations will exhibit the diffusion phenomenon (1.3). That is to say u_{tt} in (1.5) is "weaker" than u_t and $-\Delta u$ in the sense that the L^2 norm of u_{tt} decays to 0 more quickly as $t \to \infty$. Therefore, we treat u_{tt} in (1.5) as a perturbation. Hence, one might expect that the solution to (1.5) behaves like a solution to the heat equation $v_t(x,t) - \Delta v(x,t) = 0$. This insight, which was previously known, appears to be necessary in obtaining analogs to (1.6) in more general settings. We will show that this insight holds in a "weaker" sense in more general settings. This is the "new" main tool that we use.

Following Matsumura's results, many authors have considered variants of (1.5), including variants where $-\Delta$ is replaced with a more general, *time-independent* operator. Many authors have also considered variants of the form

$$u_{tt}(x,t) + a(x)b(t)u_t(x,t) - \Delta u(x,t) = 0, \quad x \in X,\ t > 0, \tag{1.7}$$

where $X \subset \mathbb{R}^N$ and the damping coefficient is in separable form. Note that until recently, (1.7) was considered with either space- or time-dependent damping coefficients, and the methods used for these two types of problems are incompatible.

Ikehata [11, 10] and Ono [30] considered (1.7) as an IBVP, initial boundary value problem, with constant damping; here the set $X = \mathbb{R}^N \setminus K$ for some K compact. Similar IBVPs with space-dependent damping coefficients have been thoroughly studied, e.g., Nakao [26], Ikehata [12, 13], and Mochizuki and Nakao [23]. Sobajima and Wakasugi [36] considered (1.7) with radially symmetric, slowly decaying space-dependent damping. They demonstrate the diffusion phenomenon, obtaining a sharp decay estimate for a modified L^2 norm of the solution.

Todorova and Yordanov [41] considered the problem (1.7) with $X = \mathbb{R}^N$, a constant damping coefficient and the nonlinear term $|u|^p$. They developed a weighted energy method that uses a special weight related to the fundamental solution of (1.5).

Todorova and Yordanov [42] considered problem (1.7) with $X = \mathbb{R}^N$ and a slowly decaying, space-dependent damping coefficient. They used a weighted energy method with a special weight. Similar weighted energy methods were implemented in [17, 19, 20, 27, 28, 36]. Nishihara and Zhai [28] and Nishihara [27] considered (1.7) with a defocusing nonlinear term $-|u|^{p-1}u$ and a slowly changing damping coefficient that was either space- or time-dependent . This was followed by Lin, Nishihara and Zhai [19, 20], and Khader [17] who studied (1.7) with a slowly changing, radially symmetric damping coefficient and a defocusing nonlinear term $-|u|^{p-1}u$.

The problem (1.7) with $X = \mathbb{R}^N$ and a slowly changing, time-dependent damping coefficient has been thoroughly investigated; see Reissig and Wirth [34], and Wirth [43, 44]. Using Fourier methods, Wirth obtained sharp $L^p(\mathbb{R}^N) - L^q(\mathbb{R}^N)$ decay estimates for solutions. Mochizuki and Nakazawa [24] showed energy decay for solutions to (1.7) with a nonseparable damping coefficient.

For damped wave equations with slowly changing, space-dependent coefficients, Radu, Todorova and Yordanov [31] proved the exact gain in the decay rate for all higher order energies in terms of the first order energy.

Many authors have demonstrated the diffusion phenomenon for the abstract problem $u_{tt} + u_t + Bu = 0$, where u belongs to a Hilbert space $\mathcal{H}$ and B is a *time-independent* nonnegative self-adjoint operator in $\mathcal{H}$. For example, see Ikehata and Nishihara [14], Chill and Haraux [2], Radu, Todorova and Yordanov [32], and Nishiyama [29].

Ikehata, Todorova and Yordanov [15] showed a more complex diffusion phenomenon for abstract wave equations with strong damping. Then Radu, Todorova and Yordanov [33] proved the diffusion phenomenon for the problem $Cu_{tt} + u_t + Bu = 0$ in a Hilbert space $\mathcal{H}$, where B and C are two noncommuting self-adjoint operator on $\mathcal{H}$, which excludes the use of the spectral theorem. Instead, they used consecutive approximations with conveniently defined diffusion solutions. They also expanded their decay gains that originated in [31], giving the exact gain in the decay rate for $\|\partial_t^n u\|$ in terms of $\|u\|$.

By resolvent arguments, Nishiyama [29] showed the diffusion phenomenon for the problem $u_{tt} + Au_t + Bu = 0$ in a Hilbert space $\mathcal{H}$. Here A and B are two noncommuting self-adjoint operator on $\mathcal{H}$, satisfying some additional conditions. Yamazaki [45] studied abstract wave equations with time-dependent damping.

Dirichlet forms have been thoroughly investigated, e.g., see the book by Fukushima, Oshima and Takeda [6].

Extensive research has been done to study the behavior of solutions to parabolic PDEs in metric measure spaces, such as (1.2), with operators related to general Dirichlet forms; for example, see Sturm [37], Lierl and Saloff-Coste [18] and the references therin. Significant results for these parabolic PDEs will be presented in chapter 3. These results allow us to contribute to what little was known about solutions to hyperbolic PDEs in metric measure spaces, such as (1.1), in the context of general Dirichlet forms.

1.5 Dissertation structure

In chapter 2, we show that the diffusion phenomenon (1.3) holds in the particular case where (X, d, m) is $N-$dimensional Euclidean space with Lebesgue measure; the operator $A(t)$ in (1.1) will be a divergence form operator defined via $A(t)f(x) := -\nabla \cdot (a(x,t)\nabla f(x))$, where $a_1 \leq a(x,t) \leq a_2$ for constants $a_1, a_2 > 0$. The work in chapter 2 primarily consists of developing three key tools to show (1.3); this work is an important stepping stone that allows us to complete the work in chapter 3.

In chapter 3, we present our main results, showing that the diffusion phenomenon (1.3) holds in more general settings via expanding on and significantly modifying the three key tools that were developed in chapter 2; the results we give are valid for more general metric measure spaces X and more general operators $A(t)$. As a consequence, we are able to show the diffusion phenomenon when $A(t)$ is not uniformly elliptic, or when $X = S^1 \times \mathbb{R}^N$, where S^1 is the unit circle in $\mathbb{R}^2$. In chapter 3 we state well-known results for parabolic PDEs, which, by themselves, do not necessarily give any decay for the solution u to (1.1). Also, fairly routine properties such as a finite speed of propagation for u require proof.

A proof of existence, uniqueness and regularity for the solution u to (1.1) is given in appendix A; this is only necessary for the work in chapter 3. Appendix B contains proofs of necessary lemmas.

Chapter 2

The diffusion phenomenon

The material in this chapter has been published in the journal *Discrete & Continuous Dynamical Systems*; see Taylor [39].

Let (X, d, m) be $N-$dimensional Euclidean space with Lebesgue measure. In this chapter, we study a particular case of problem (1.1), namely

$$\begin{cases} u_{tt}(x,t) + u_t(x,t) - \nabla \cdot \big(a(x,t)\nabla u(x,t)\big) = 0, & x \in \mathbb{R}^N,\ t > 0, \\ (u, u_t)(x, 0) = (u_0, u_1)(x), & x \in \mathbb{R}^N. \end{cases} \tag{2.1}$$

We prove that the solution u to (2.1) exhibits the diffusion phenomenon (1.3), meaning that u asymptotically behaves like a solution to

$$\begin{cases} v_t(x,t) - \nabla \cdot \big(a(x,t)\nabla v(x,t)\big) = 0, & x \in \mathbb{R}^N,\ t > 0, \\ v(x, 0) = u_0(x) + u_1(x), & x \in \mathbb{R}^N. \end{cases} \tag{2.2}$$

As a corollary to the diffusion phenomenon, we obtain a sharp decay estimate for $\|u(x,t)\|_{L^2(\mathbb{R}^N)}$. We also obtain decay estimates for $\|\nabla u(x,t)\|_{L^2(\mathbb{R}^N)}$ and $\|u_t(x,t)\|_{L^2(\mathbb{R}^N)}$.

We introduce three key tools to show the diffusion phenomenon (1.3). The first tool is the *improved decay*, which specifically refers to the gains in the decay rates for space and time derivatives of u in terms of u. This gain in decay is expressed in a weighted average sense. Note that the improved decay was initially developed in Radu, Todorova and Yordanov [31, 33].

The *weighted energy method* that was developed in Todorova and Yordanov [41] is the second key tool. One important consequence of this weighted energy method is that the solution u to (2.1) decays exponentially for x outside of the ball $\{x \in \mathbb{R}^N : |x| < (t+1)^{(1+\delta)/2}\}$, where $\delta > 0$ can be arbitrarily small.

The third key tool involves the *fundamental solution* of (2.2), which encodes decay properties for (2.2); these properties are found in Friedman [5, Chapter 1]. We prove that $u - v$, the difference between the solutions of (2.1) and (2.2), can be expressed in terms of the fundamental solution of (2.2) acting on derivatives of u. This representation of $u - v$ permits the three key tools to work together.

2.1 Assumptions, basic facts and results

2.1.1 Assumptions for problem 2.1

For $k \in \mathbb{N}$, let $H^k(\mathbb{R}^N)$ be the standard Sobolev space $W^{k,2}(\mathbb{R}^N)$. We assume the data (u_0, u_1) are in $H^3\left(\mathbb{R}^N\right) \times H^2\left(\mathbb{R}^N\right)$, and the support of the data satisfies $supp(u_0, u_1) \subset \{x \in \mathbb{R}^N : |x| < R_0\}$ for some $R_0 > 0$. We also assume that $a(x,t) \in C^2\left(\mathbb{R}^{N+1}\right)$ and its first and second order derivatives are bounded and continuous in $\mathbb{R}^{N+1}$; this includes the mixed space-time derivatives. In addition, we assume

$$a_1 \le a(x,t) \le a_2, \tag{B1}$$

$$|a_t(x,t)| \le a_3 \frac{a(x,t)}{t+1}, \tag{B2}$$

$$|a_{tt}(x,t)| \le a_4 \frac{a(x,t)}{(t+1)^2}, \tag{B3}$$

where the constants a_1, a_2, a_3, and $a_4 > 0$.

Remark 2.1. *Whenever assumption (B1) is satisfied, so is assumption (D) with $c_1 = a_1$ and $c_2 = a_2$. The reference Dirichlet form $\mathcal{E}$ is defined via $\mathcal{E}(f,g) := \int_{\mathbb{R}^N} \nabla f(x) \cdot \nabla g(x) dx$, where $f, g \in H^1(\mathbb{R}^N)$. Similarly, the Dirichlet form $\mathcal{E}_t$ is defined via $\mathcal{E}_t(f,g) := \int_{\mathbb{R}^N} a(x,t) \nabla f(x) \cdot \nabla g(x) dx$.*

Remark 2.2. *With the reference Dirichlet form $\mathcal{E}$ defined in remark 2.1, we see that the intrinsic metric ρ defined in chapter 1 is the Euclidean metric d.*

Remark 2.3. *Generalized versions of assumptions* (B2) *and* (B3) *will appear in chapter 3.*

2.1.2 Existence, uniqueness and regularity for problem (2.1)

These are given by

Lemma 2.4. *(Existence, uniqueness and regularity) Under the assumptions of subsection 2.1.1, problem (2.1) admits a unique solution such that*

$$u \in \bigcap_{i=0}^{3} \mathcal{C}^i([0,\infty); H^{3-i}(\mathbb{R}^N)); \tag{2.3}$$

see Ikawa [8, Theorem 2].

2.1.3 Results for problem (2.1)

Theorem 2.5. *(Diffusion phenomenon) Let $u(x,t)$ be the solution to (2.1), where the assumptions in subsection 2.1.1 hold. Then for $t \geq 0$*

$$\|u(x,t) - v(x,t)\|^2_{L^2(\mathbb{R}^N)} \leq C\ (t+1)^{-\frac{N+2}{2}}\ \|(u_0,u_1)\|^2_{H^2\times H^1(\mathbb{R}^N)},$$

where $v(x,t)$ is a prescribed solution to (2.2), and the constant C depends on $a_1,\ldots,a_4,N$, and R_0.

The prescribed solution to (2.2) will be shown to have the property $\|v(x,t)\|^2_{L^2(\mathbb{R}^N)} \leq C\ (t+1)^{-\frac{N}{2}}\ \|(u_0,u_1)\|^2_{H^2\times H^1(\mathbb{R}^N)}$. Combining this with Theorem 2.5 gives the following corollary.

Corollary 2.6. *Let $u(x,t)$ be the solution to (2.1), where the assumptions in subsection 2.1.1 hold. Then for $t \geq 0$, the following hold:*

$$\|u(x,t)\|^2_{L^2(\mathbb{R}^N)} \leq C\ (t+1)^{-\frac{N}{2}}\ \|(u_0,u_1)\|^2_{H^2\times H^1(\mathbb{R}^N)}\,, \tag{i}$$

$$\|\nabla u(x,t)\|^2_{L^2(\mathbb{R}^N)} \leq C\ \ln(t+2)\,(t+1)^{-\frac{N}{2}-1}\ \|(u_0,u_1)\|^2_{H^2\times H^1(\mathbb{R}^N)}\,, \tag{ii}$$

$$\|u_t(x,t)\|^2_{L^2(\mathbb{R}^N)} \leq C\ \ln(t+2)\,(t+1)^{-\frac{N}{2}-2}\ \|(u_0,u_1)\|^2_{H^2\times H^1(\mathbb{R}^N)}\,, \tag{iii}$$

where the constant C depends on $a_1,\ldots,a_4, N$, and R_0.

Remark 2.7. *Using the same method, it is possible to consider a more general problem*

$$\begin{cases} c(x,t)\,u_{tt} + b(x,t)\,u_t - \nabla\cdot\big(a(x,t)\nabla u\big) = 0, & x\in\mathbb{R}^N,\ t>0,\\ (u,u_t)(x,0) = (u_0,u_1)(x), & x\in\mathbb{R}^N, \end{cases}$$

where $a,b,c\in C^2\left(\mathbb{R}^{N+1}\right)$ have bounded first and second order derivatives and each satisfies assumptions similar to B1, B2, and B3. Conclusions analogous to Theorem 2.5 and Corollary 2.6 can be achieved via analogous proofs.

Remark 2.8. *In this chapter, we have estimates for spatial derivatives of the fundamental solution to (2.2); see Lemma 2.21. In chapter 3, we are missing analogs to those estimates, and as a consequence, the analog in chapter 3 to the diffusion phenomenon Theorem 2.5 is not as strong as it potentially could be.*

2.2 Improved decay for dissipative wave equations

2.2.1 Preliminary lemmas

The lemmas in this subsection are standard and presented for completeness.

Lemma 2.9. *(Energy inequality) Let $u(x,t)$ be the solution to (2.1), where the assumptions in subsection 2.1.1 hold, and let $T>0$. Then for $0\leq t\leq T$*

$$\|u(x,t)\|^2_{H^2(\mathbb{R}^N)} + \|u_t(x,t)\|^2_{H^1(\mathbb{R}^N)} + \|u_{tt}(x,t)\|^2_{L^2(\mathbb{R}^N)} \leq C(T)\,\|(u_0,u_1)\|^2_{H^2\times H^1(\mathbb{R}^N)}\,;$$

see Ikawa [8, Proposition 2.6] or cf. [9, Theorem 2.15].

Lemma 2.10. *(Finite speed of propagation) Let $u(x,t)$ be the solution to (2.1), where the assumptions in subsection 2.1.1 hold. Then u has a finite speed of propagation; see Ikawa [9, Theorem 2.7]. Hence, the support of u is compact for all $t>0$ since the support of the initial data (u_0,u_1) is compact.*

2.2.2 Improved decay

The purpose of this subsection is to obtain the gains in the decay rates for derivatives of u in terms of u. These gains in decay are expressed in a weighted average sense.

Definition 2.11. *For $i=1,2$, respectively, we use the first and second energies:*

$$E_i(t;u) := \frac{1}{2}\int_{\mathbb{R}^N} \left(\partial_t^i u(x,t)\right)^2 + a(x,t)|\nabla\partial_t^{i-1}u(x,t)|^2 dx.$$

The improved decay for the first energy $E_1(t;u)$ is proved in the following proposition.

Proposition 2.12. *Let $u(x,t)$ be the solution to (2.1), and let the assumptions in subsection 2.1.1 be satisfied. For $r\geq 0$ and $\theta\geq 0$,*

$$\int_0^r (t+1)^\theta E_1(t;u)dt \leq C\left\|(u_0,u_1)\right\|^2_{H^2\times H^1(\mathbb{R}^N)} + C\int_0^r (t+1)^{\theta-1}\left\|u\right\|^2_{L^2(\mathbb{R}^N)}dt, \tag{2.4}$$

where C depends on a_2, a_3, and θ.

Proof. We begin by taking the $L^2_x(\mathbb{R}^N)$ inner product of equation (2.1) and $2u_t$. Then apply assumption (B2) and get

$$\partial_t\left(\left\|u_t\right\|^2_{L^2(\mathbb{R}^N)} + \langle a\nabla u\,,\,\nabla u\rangle_{L^2(\mathbb{R}^N)}\right) \leq -2\left\|u_t\right\|^2_{L^2(\mathbb{R}^N)} + \frac{a_3}{t+1}\langle a\nabla u\,,\,\nabla u\rangle_{L^2(\mathbb{R}^N)}. \tag{2.5}$$

Similarly, we take the $L^2_x(\mathbb{R}^N)$ inner product of equation (2.1) and u to obtain

$$\partial_t\left(\langle u_t\,,\,u\rangle_{L^2(\mathbb{R}^N)} + \frac{1}{2}\left\|u\right\|^2_{L^2(\mathbb{R}^N)}\right) = \left\|u_t\right\|^2_{L^2(\mathbb{R}^N)} - \langle a\nabla u\,,\,\nabla u\rangle_{L^2(\mathbb{R}^N)}. \tag{2.6}$$

Next, define the continuously differentiable function

$$Y(t) := \|u_t\|^2_{L^2(\mathbb{R}^N)} + \langle u_t \,,\, u\rangle_{L^2(\mathbb{R}^N)} + \frac{1}{2}\|u\|^2_{L^2(\mathbb{R}^N)} + \langle a\nabla u \,,\, \nabla u\rangle_{L^2(\mathbb{R}^N)}\,;$$

for the regularity, see Lemma 2.4. Then combine (2.5) with (2.6) and add $\frac{\theta}{t+1}Y(t)$ to both sides. This gives

$$\frac{\theta}{t+1}Y(t) + Y'(t) + E_1(t;u) \leq \frac{\theta}{t+1}Y(t) + \frac{a_3}{t+1}\langle a\nabla u \,,\, \nabla u\rangle_{L^2(\mathbb{R}^N)} - E_1(t;u). \tag{2.7}$$

Notice that $u_t u \geq -\frac{1}{2}(u^2+u_t^2)$, giving

$$0 \leq Y(t). \tag{2.8}$$

Similarly,

$$Y(t) \leq \|u\|^2_{L^2(\mathbb{R}^N)} + 3E_1(t;u), \tag{2.9}$$

since $u_t u \leq \frac{1}{2}(u^2+u_t^2)$ and $2E_1(t;u) = \|u_t\|^2_{L^2(\mathbb{R}^N)} + \langle a\nabla u \,,\, \nabla u\rangle_{L^2(\mathbb{R}^N)}$. Apply (2.9) and $\langle a\nabla u \,,\, \nabla u\rangle_{L^2(\mathbb{R}^N)} \leq 2E_1(t;u)$ to the RHS of (2.7) and obtain

$$\frac{\theta}{t+1}Y(t) + Y'(t) + E_1(t;u) \leq \frac{\theta}{t+1}\|u\|^2_{L^2(\mathbb{R}^N)} + \left(\frac{3\theta+2a_3}{t+1} - 1\right)E_1(t;u). \tag{2.10}$$

Multiply both sides of (2.10) by the integrating factor $(t+1)^\theta$ to see that

$$\begin{aligned} &\partial_t\left((t+1)^\theta Y(t)\right) + (t+1)^\theta E_1(t;u) \\ \leq&\theta(t+1)^{\theta-1}\|u\|^2_{L^2(\mathbb{R}^N)} + (t+1)^\theta\left(\frac{3\theta+2a_3}{t+1} - 1\right)E_1(t;u). \end{aligned} \tag{2.11}$$

Next integrate both sides of (2.11) with respect to t, from 0 to r. To complete the proof, we estimate the integrals of the first and last terms of (2.11) by the initial data. Note that

(2.8) and (2.9), followed by assumption (B1) give

$$\begin{aligned}(t+1)^\theta Y(t)\Big|_{t=0}^{r} &= (r+1)^\theta Y(r) - Y(0)\\ &\geq 0 - \left(\|u_0\|^2_{L^2(\mathbb{R}^N)} + 3E_1(0;u)\right)\\ &\geq -C(a_2)\,\|(u_0,u_1)\|^2_{H^2\times H^1(\mathbb{R}^N)}\,.\end{aligned}$$

Define $T_0 := \max\{0, 3\theta + 2a_3 - 1\}$. Then for all $r \geq 0$,

$$\int_0^r (t+1)^\theta \left(\frac{3\theta+2a_3}{t+1} - 1\right) E_1(t;u)dt \leq \int_0^{T_0} (t+1)^\theta \left(\frac{3\theta+2a_3}{t+1} - 1\right) E_1(t;u)dt, \quad (2.12)$$

since $\frac{3\theta+2a_3}{t+1} - 1 \leq 0$ for $t \geq T_0$. Apply assumption (B1) and then the energy inequality Lemma 2.9 to the RHS of (2.12), obtaining

$$\int_0^r (t+1)^\theta \left(\frac{3\theta+2a_3}{t+1} - 1\right) E_1(t;u)dt \leq C(a_2,a_3,\theta)\,\|(u_0,u_1)\|^2_{H^2\times H^1(\mathbb{R}^N)}$$

for all $r \geq 0$. Therefore, the proof of (2.4) is complete. □

The improved decay for $\|u_t\|^2_{L^2(\mathbb{R}^N)}$ is shown in the following proposition.

Proposition 2.13. *Let $u(x,t)$ be the solution to (2.1), and let the assumptions in subsection 2.1.1 be satisfied. For $r \geq 0$ and $\theta \geq 0$,*

$$\int_0^r (t+1)^{\theta+1}\,\|u_t\|^2_{L^2(\mathbb{R}^N)}\,dt \leq C\,\|(u_0,u_1)\|^2_{H^1\times L^2(\mathbb{R}^N)} + C\int_0^r (t+1)^{\theta-1}\,\|u\|^2_{L^2(\mathbb{R}^N)}\,dt, \quad (2.13)$$

where C depends on $a_2, a_3,$ and θ.

Proof. Add $\frac{\theta+1}{t+1}E_1(t;u)$ to both sides of (2.5) to obtain

$$\frac{\theta+1}{t+1}E_1(t;u) + \partial_t\,E_1(t;u) \leq -2\,\|u_t\|^2_{L^2(\mathbb{R}^N)} + \frac{\theta+1}{t+1}E_1(t;u) + \frac{a_3}{t+1}\,\langle a\nabla u\,,\,\nabla u\rangle_{L^2(\mathbb{R}^N)}\,.$$

Next, bound $\langle a\nabla u\,,\,\nabla u\rangle_{L^2(\mathbb{R}^N)}$ from above by $2E_1(t;u)$, and then multiply both sides of the resulting inequality by the integrating factor $(t+1)^{\theta+1}$. This gives

$$\partial_t\,\left((t+1)^{\theta+1}E_1(t;u)\right) \leq -2(t+1)^{\theta+1}\,\|u_t\|^2_{L^2(\mathbb{R}^N)} + (\theta+1+2a_3)(t+1)^\theta E_1(t;u).$$

Next, integrate both sides of this inequality with respect to t, from 0 to r, and note that

$$(t+1)^{\theta+1}E_1(t;u)\big|_{t=0}^{r} \geq -E_1(0;u).$$

To complete the proof of (2.13), apply the improved decay Proposition 2.12 to the term $\int_0^r (\theta+1+2a_3)(t+1)^{\theta}E_1(t;u)dt$, obtaining the last term on the RHS of (2.13). □

The next two propositions show the improved decay for $E_2(t;u)$ and $\|u_{tt}\|^2_{L^2(\mathbb{R}^N)}$, respectively. These propositions are analogous to Propositions 2.12 and 2.13, except with larger weights on their left-hand sides.

Proposition 2.14. *Let $u(x,t)$ be the solution to (2.1), and let the assumptions in subsection 2.1.1 be satisfied. For $r \geq 0$ and $\theta \geq 0$,*

$$\int_0^r (t+1)^{\theta+2}E_2(t;u)dt \leq C\,\|(u_0,u_1)\|^2_{H^2\times H^1(\mathbb{R}^N)} + C\int_0^r (t+1)^{\theta-1}\,\|u\|^2_{L^2(\mathbb{R}^N)}\,dt, \tag{2.14}$$

where C depends on $a_2, a_3, a_4,$ and θ.

Proof. Begin by taking the $L^2_x(\mathbb{R}^N)$ inner product of $\partial_t\left(u_{tt}+u_t-\nabla\cdot\big(a\nabla u\big)\right)=0$ and $2u_{tt}$ to obtain

$$\begin{aligned}&\partial_t\Big(\,\|u_{tt}\|^2_{L^2(\mathbb{R}^N)} + \langle a\nabla u_t\,,\ \nabla u_t\rangle_{L^2(\mathbb{R}^N)} + 2\,\langle a_t\nabla u\,,\ \nabla u_t\rangle_{L^2(\mathbb{R}^N)}\Big)\\ &= -\,2\,\|u_{tt}\|^2_{L^2(\mathbb{R}^N)} + \langle 3a_t\nabla u_t + 2a_{tt}\nabla u\,,\ \nabla u_t\rangle_{L^2(\mathbb{R}^N)}\,.\end{aligned} \tag{2.15}$$

Similarly we take the $L^2_x(\mathbb{R}^N)$ inner product of $\partial_t\left(u_{tt}+u_t-\nabla\cdot\big(a\nabla u\big)\right)=0$ and u_t to obtain

$$\begin{aligned}&\partial_t\left(\langle u_{tt}\,,\ u_t\rangle_{L^2(\mathbb{R}^N)} + \frac{1}{2}\,\|u_t\|^2_{L^2(\mathbb{R}^N)}\right)\\ &= \|u_{tt}\|^2_{L^2(\mathbb{R}^N)} - \langle a\nabla u_t\,,\ \nabla u_t\rangle_{L^2(\mathbb{R}^N)} - \langle a_t\nabla u\,,\ \nabla u_t\rangle_{L^2(\mathbb{R}^N)}\,.\end{aligned} \tag{2.16}$$

Now, define the functions

$$Z_1(t) := \frac{(a_3)^2}{(t+1)^2} \langle a\nabla u \,,\, \nabla u\rangle_{L^2(\mathbb{R}^N)} \,, \;\; Z_2(t) := \langle a_t\nabla u_t \,,\, \nabla u_t\rangle_{L^2(\mathbb{R}^N)} \,, \;\text{ and}$$

$$Z_3(t) := \langle (2a_{tt} - a_t)\, \nabla u \,,\, \nabla u_t\rangle_{L^2(\mathbb{R}^N)} \,.$$

Next, define the continuously differentiable function

$$Y(t) := \|u_{tt}\|^2_{L^2(\mathbb{R}^N)} + \langle u_{tt} \,,\, u_t\rangle_{L^2(\mathbb{R}^N)} + \frac{1}{2}\|u_t\|^2_{L^2(\mathbb{R}^N)}$$
$$+ \langle a\nabla u_t \,,\, \nabla u_t\rangle_{L^2(\mathbb{R}^N)} + 2\langle a_t\nabla u \,,\, \nabla u_t\rangle_{L^2(\mathbb{R}^N)} + Z_1(t),$$

noting that $Z_1(t)$ is also continuously differentiable. Importantly, the presence of $Z_1(t)$ in $Y(t)$ ensures that $Y(t) \geq 0$, which will be shown later.

Observe that the left-hand sides of (2.15) and (2.16) sum to $Y'(t) - Z_1'(t)$. Thus (2.15) combined with (2.16) gives $Y'(t) + 2E_2(t;u) = Z_1'(t) + 3Z_2(t) + Z_3(t)$, and adding $\frac{\theta+2}{t+1}Y(t)$ to both sides gives

$$\frac{\theta+2}{t+1}Y(t) + Y'(t) + 2E_2(t;u) = \frac{\theta+2}{t+1}Y(t) + Z_1'(t) + 3Z_2(t) + Z_3(t). \tag{2.17}$$

To estimate the RHS of (2.17) from above, we estimate $Z_1'(t)$, $Z_2(t)$, and $Z_3(t)$. First, notice that

$$Z_1'(t) \leq \frac{a_3-1}{t+1}Z_1(t) + \frac{(a_3)^2}{t+1}\langle a\nabla u_t \,,\, \nabla u_t\rangle_{L^2(\mathbb{R}^N)} \,, \tag{2.18}$$

by assumption (B2) and $2a\,|\nabla u|\,|\nabla u_t| \leq \frac{a}{t+1}|\nabla u|^2 + (t+1)a\,|\nabla u_t|^2$. Next, assumption (B2) gives

$$Z_2(t) \leq \frac{a_3}{t+1}\langle a\nabla u_t \,,\, \nabla u_t\rangle_{L^2(\mathbb{R}^N)} \,. \tag{2.19}$$

Observe

$$Z_3(t) \leq \left(\frac{4(a_4)^2}{(a_3)^2(t+1)^2} + 1\right) Z_1(t) + \frac{1}{2}\langle a\nabla u_t \,,\, \nabla u_t\rangle_{L^2(\mathbb{R}^N)} \,, \tag{2.20}$$

via $|(2a_{tt} - a_t)\,\nabla u|\,|\nabla u_t| \leq \frac{(2a_{tt}-a_t)^2|\nabla u|^2}{2a} + \frac{a|\nabla u_t|^2}{2}$, with $(2a_{tt} - a_t)^2 \leq \frac{8(a_4)^2}{(t+1)^4}a^2 + \frac{2(a_3)^2}{(t+1)^2}a^2$ by assumptions (B2) and (B3). Now use (2.18) - (2.20) to estimate the RHS of (2.17) from above by

$$\frac{\theta+2}{t+1}Y(t)+C\left(a_3,a_4\right)Z_1(t)+\left(\frac{C\left(a_3\right)}{t+1}+\frac{1}{2}\right)\left\langle a\nabla u_t\;,\;\nabla u_t\right\rangle_{L^2(\mathbb{R}^N)}. \tag{2.21}$$

The following estimates hold:

$$Z_1(t)\le\frac{2(a_3)^2}{(t+1)^2}E_1(t;u), \tag{2.22}$$

$$0\le Y(t), \tag{2.23}$$

$$Y(t)\le\|u_t\|^2_{L^2(\mathbb{R}^N)}+\frac{3}{2}\|u_{tt}\|^2_{L^2(\mathbb{R}^N)}+2\left\langle a\nabla u_t\;,\;\nabla u_t\right\rangle_{L^2(\mathbb{R}^N)}+2Z_1(t). \tag{2.24}$$

The proof of (2.22) follows from the definitions of $Z_1(t)$ and $E_1(t;u)$. To show (2.23), note $Y(t)\ge\left\langle a\nabla u_t\;,\;\nabla u_t\right\rangle_{L^2(\mathbb{R}^N)}+2\left\langle a_t\nabla u\;,\;\nabla u_t\right\rangle_{L^2(\mathbb{R}^N)}+Z_1(t)$ since $u_{tt}u_t\ge-\frac{1}{2}\left(u_t^2+u_{tt}^2\right)$. Also, $-2a_t\nabla u\cdot\nabla u_t\le\frac{(a_t)^2|\nabla u|^2}{a}+a\left|\nabla u_t\right|^2$, and $(a_t)^2\le\frac{(a_3)^2a^2}{(t+1)^2}$ by assumption (B2). Hence $-2a_t\nabla u\cdot\nabla u_t\le\frac{(a_3)^2}{(t+1)^2}a\left|\nabla u\right|^2+a\left|\nabla u_t\right|^2$, meaning that $0\le Y(t)$. The proof of (2.24) is similar to the proof of (2.23).

Apply (2.22) and (2.24) to bound (2.21), and hence the RHS of (2.17), from above by

$$\begin{aligned}&\frac{\theta+2}{t+1}\|u_t\|^2_{L^2(\mathbb{R}^N)}+\frac{C\left(a_3,a_4,\theta\right)}{(t+1)^2}E_1(t;u)\\&+\frac{1}{2}\left(\frac{C\left(a_3,\theta\right)}{t+1}+1\right)\left(\left\langle a\nabla u_t\;,\;\nabla u_t\right\rangle_{L^2(\mathbb{R}^N)}+\|u_{tt}\|^2_{L^2(\mathbb{R}^N)}\right).\end{aligned} \tag{2.25}$$

Replace the RHS of (2.17) with (2.25) to obtain

$$\begin{aligned}&\frac{\theta+2}{t+1}Y(t)+Y'(t)+\frac{1}{2}E_2(t;u)\\\le&\frac{\theta+2}{t+1}\|u_t\|^2_{L^2(\mathbb{R}^N)}+\frac{C\left(a_3,a_4,\theta\right)}{(t+1)^2}E_1(t;u)+\left(\frac{C\left(a_3,\theta\right)}{t+1}-\frac{1}{2}\right)E_2(t;u),\end{aligned} \tag{2.26}$$

recalling that $\left\langle a\nabla u_t\;,\;\nabla u_t\right\rangle_{L^2(\mathbb{R}^N)}+\|u_{tt}\|^2_{L^2(\mathbb{R}^N)}=2E_2(t;u)$. Multiply both sides of (2.26) by the integrating factor $(t+1)^{\theta+2}$ and integrate in t, from 0 to r. Then apply the improved decay Propositions 2.12 and 2.13 to the integrals involving $E_1(t;u)$ and $\|u_t\|^2_{L^2(\mathbb{R}^N)}$, respectively.

This way, we get the last term on the RHS of (2.14). Thus we only need to bound the integrals involving the first two terms and the last term of (2.26) by the initial data.

The integral involving the first two terms of (2.26) is bounded via inequalities (2.23) and (2.24), i.e., observe that

$$\begin{aligned}\int_0^r (t+1)^{\theta+2}\left(\frac{\theta+2}{t+1}Y(t)+Y'(t)\right)dt &= (t+1)^{\theta+2}Y(t)\big|_{t=0}^{r}\\ &\geq -C(a_2,a_3)\left\|(u_0,u_1)\right\|^2_{H^2\times H^1(\mathbb{R}^N)}.\end{aligned}$$

To bound the integral involving the last term of (2.26), define $T_0 := \max\{0, 2C(a_3,\theta)-1\}$. Then for all $r\geq 0$,

$$\begin{aligned}&\int_0^r (t+1)^{\theta+2}\left(\frac{C(a_3,\theta)}{t+1}-\frac{1}{2}\right)E_2(t;u)dt\\ \leq &\int_0^{T_0} (t+1)^{\theta+2}\left(\frac{C(a_3,\theta)}{t+1}-\frac{1}{2}\right)E_2(t;u)dt,\end{aligned}\tag{2.27}$$

since $\frac{C(a_3,\theta)}{t+1}-\frac{1}{2}\leq 0$ for $t\geq T_0$. To complete the proof, apply assumption (B1) and then the energy inequality Lemma 2.9 to the RHS of (2.27), obtaining

$$\int_0^r (t+1)^{\theta+2}\left(\frac{C(a_3,\theta)}{t+1}-\frac{1}{2}\right)E_2(t;u)dt\leq C(a_2,a_3,\theta)\left\|(u_0,u_1)\right\|^2_{H^2\times H^1(\mathbb{R}^N)}$$

for all $r\geq 0$. □

Proposition 2.15. *Let $u(x,t)$ be the solution to (2.1), and let the assumptions in subsection 2.1.1 be satisfied. For $r\geq 0$ and $\theta\geq 0$,*

$$\begin{aligned}\int_0^r (t+1)^{\theta+3}\left\|u_{tt}\right\|^2_{L^2(\mathbb{R}^N)}dt \leq& C\left\|(u_0,u_1)\right\|^2_{H^2\times H^1(\mathbb{R}^N)}\\ &+C\int_0^r (t+1)^{\theta-1}\left\|u\right\|^2_{L^2(\mathbb{R}^N)}dt,\end{aligned}\tag{2.28}$$

where C depends on a_2, a_3, a_4, and θ.

Proof. We use the functions $Z_1(t)$ and $Z_2(t)$ defined in the improved decay Proposition 2.14. Define the functions

$$Z_4(t) := 2\left\langle a_{tt}\nabla u\ ,\ \nabla u_t\right\rangle_{L^2(\mathbb{R}^N)} \text{ and}$$
$$Y(t) := \|u_{tt}\|^2_{L^2(\mathbb{R}^N)} + \left\langle a\nabla u_t\ ,\ \nabla u_t\right\rangle_{L^2(\mathbb{R}^N)} + 2\left\langle a_t\nabla u\ ,\ \nabla u_t\right\rangle_{L^2(\mathbb{R}^N)} + Z_1(t).$$

Notice that (2.15) gives $Y'(t) + 2\|u_{tt}\|^2_{L^2(\mathbb{R}^N)} = Z_1'(t) + 3Z_2(t) + Z_4(t)$, and adding $\frac{\theta+3}{t+1}Y(t)$ to both sides gives

$$\frac{\theta+3}{t+1}Y(t) + Y'(t) + 2\|u_{tt}\|^2_{L^2(\mathbb{R}^N)} = \frac{\theta+3}{t+1}Y(t) + Z_1'(t) + 3Z_2(t) + Z_4(t). \tag{2.29}$$

Similar to (2.20), (2.23) and (2.24), respectively, we have

$$Z_4(t) \le \frac{(a_4)^2}{(a_3)^2(t+1)}Z_1(t) + \frac{\left\langle a\nabla u_t\ ,\ \nabla u_t\right\rangle_{L^2(\mathbb{R}^N)}}{t+1}, \tag{2.30}$$

$$0 \le Y(t), \tag{2.31}$$

$$Y(t) \le \|u_{tt}\|^2_{L^2(\mathbb{R}^N)} + 2\left\langle a\nabla u_t\ ,\ \nabla u_t\right\rangle_{L^2(\mathbb{R}^N)} + 2Z_1(t). \tag{2.32}$$

To show (2.30), use $2|a_{tt}\nabla u|\,|\nabla u_t| \le \frac{(t+1)a_{tt}^2|\nabla u|^2}{a} + \frac{a|\nabla u_t|^2}{t+1}$, with $a_{tt}^2 \le \frac{(a_4)^2}{(t+1)^4}a^2$ by assumption (B3). Next, apply (2.18), (2.19), (2.22), (2.30) and (2.32) to the RHS of (2.29) and get

$$\frac{\theta+3}{t+1}Y(t) + Y'(t) + 2\|u_{tt}\|^2_{L^2(\mathbb{R}^N)} \le \frac{C\left(a_3, a_4, \theta\right)}{(t+1)^3}E_1(t;u) + \frac{C\left(a_3, \theta\right)}{t+1}E_2(t;u).$$

Then multiply both sides of this inequality by the integrating factor $(t+1)^{\theta+3}$ and integrate with respect to t, from 0 to r. To obtain the last term on the RHS of (2.28), apply the improved decay Propositions 2.12 and 2.14 to the integrals involving $E_1(t;u)$ and $E_2(t;u)$, respectively. Note that inequalities (2.31) and (2.32) give

$$\begin{aligned}\int_0^r (t+1)^{\theta+3}\left(\frac{\theta+2}{t+1}Y(t) + Y'(t)\right)dt &= (t+1)^{\theta+3}Y(t)\big|_{t=0}^{r}\\ &\ge -C(a_2, a_3)\left\|(u_0, u_1)\right\|^2_{H^2\times H^1(\mathbb{R}^N)}.\end{aligned}$$

Therefore, the proof is complete. $\square$

2.3 Weighted energy method

Let $\delta > 0$ and $\Omega(t) := \{x \in \mathbb{R}^N : |x| \geq (t+1)^{(1+\delta)/2}\}$. One goal of this section is to prove that the derivatives of the solution to (2.1) decay exponentially for $x \in \Omega(t)$. More precisely,

$$\|\partial_t^m \nabla^n u\|^2_{L^1(\Omega(t))} \leq C\ e^{-k(t+1)^\delta}\ \|(u_0, u_1)\|^2_{H^2 \times H^1(\mathbb{R}^N)}$$

for some $k > 0$, where $n = 0, 1$ and $m = 1 - n, 2 - n$; see Proposition 2.20.

Definition 2.16. *The weight we use in this section is*

$$W(x,t) := e^{\gamma \frac{|x|^2}{t+1}}, \ \textit{where } \gamma > 0 \textit{ will be chosen conveniently.} \tag{2.33}$$

The following observations will be employed later:

$$0 \leq -W_t \leq \frac{W^2}{t+1}, \tag{2.34}$$

$$|\nabla W|^2 = -4\gamma W W_t. \tag{2.35}$$

Note that $-W_t \leq \frac{W^2}{t+1}$ because $m \leq e^m$ for all $m \in \mathbb{R}$.

Definition 2.17. *For $i = 1, 2$, respectively, we define the first and second weighted energies:*

$$E_{i,W}(t;u) := \int_{\mathbb{R}^N} W(x,t) \left(\left(\partial_t^i u(x,t)\right)^2 + a(x,t)|\nabla \partial_t^{i-1} u(x,t)|^2 \right) dx.$$

The first weighted energy estimate is given by the following proposition.

Proposition 2.18. *Let $u(x,t)$ be the solution to (2.1), and let the assumptions in subsection 2.1.1 be satisfied. Assume γ in (2.33) is such that $0 < \gamma \leq \frac{1}{2a_2}$. Then for $t \geq 0$,*

$$E_{1,W}(t;u) \leq (t+1)^{a_3}\, E_{1,W}(0;u). \tag{2.36}$$

Proof. For $W(x,t)$ as in (2.33), define the functions

$$Z_1(t) := 2\nabla \cdot \big(au_t W \nabla u\big) \;\text{ and }\; Z_2(t) := a\,W_t\,|\nabla u|^2 - 2W\,u_t^2 - 2u_t\,a\,\nabla W \cdot \nabla u.$$

Also define the function

$$Y(t) := W u_t^2 + aW|\nabla u|^2,$$

and note that $Y(t)$ is continuously differentiable as an $L^1\left(\mathbb{R}^N\right)$ function by Lemma 2.4. Next, multiply equation (2.1) by $2Wu_t$ to get

$$Y'(t) = Z_1(t) + Z_2(t) + a_t\,W\,|\nabla u|^2 + W_t\,u_t^2. \tag{2.37}$$

Note that $Z_2(t) \le 0$; to see this, observe that

$$-2\,(u_t)\,(a\,\nabla W \cdot \nabla u) \le 2W u_t^2 + \frac{a^2\,|\nabla W|^2 |\nabla u|^2}{2W} = 2W u_t^2 - (2a\gamma)\,a\,W_t |\nabla u|^2$$

via Young's inequality. Then recall that $-W_t \ge 0$ by (2.34), and notice that $2a\gamma \le 1$ since $a \le a_2$ and $0 < \gamma \le \frac{1}{2a_2}$.

Now using $Z_2(t) \le 0$, assumption (B2) and $W_t \le 0$, we refine (2.37) and get

$$Y'(t) \le Z_1(t) + \frac{a_3}{t+1} a\,W\,|\nabla u|^2.$$

Thus $Y'(t) \le Z_1(t) + \frac{a_3}{t+1} Y(t)$, since $a\,W\,|\nabla u|^2 \le Y(t)$, and integrating the former inequality with respect to x, in $\mathbb{R}^N$ gives

$$\partial_t E_{1,W}(t;u) \le \int_{\mathbb{R}^N} Z_1(t) dx + \frac{a_3}{t+1} E_{1,W}(t;u).$$

Now, $\int_{\mathbb{R}^N} Z_1(t) dx = 0$ since u has compact support in x, and we thus have

$$\partial_t E_{1,W}(t;u) \le \frac{a_3}{t+1} E_{1,W}(t;u).$$

To complete the proof, multiply both side of this inequality by the integrating factor $(t+1)^{-a_3}$, and then integrate with respect to t, on $[0,r]$. □

The following proposition gives the second weighted energy estimate.

Proposition 2.19. *Let $u(x,t)$ be the solution to (2.1), and let the assumptions in subsection 2.1.1 be satisfied. Assume γ in (2.33) is such that $0<\gamma\le \frac{1}{6a_2}$. Then for $t\ge 0$,*

$$E_{2,W}(t;u)\le C(a_3,a_4)\ (t+1)^{4a_3}\left(E_{1,W}(0;u)+E_{2,W}(0;u)\right). \tag{2.38}$$

Proof. For $W(x,t)$ as in (2.33), define the functions

$$Z_1(t):=2\nabla\cdot\left(\partial_t\big(a\nabla u\big)W\,u_{tt}\right),\ \ Z_2(t):=aW_t|\nabla u_t|^2-2W\,u_{tt}^2-2u_{tt}\,a\,\nabla W\cdot\nabla u_t,$$
$$Z_3(t):=a_t\nabla W\cdot\nabla u,\ \text{ and }\ Z_4(t):=\left(a_{tt}W+a_tW_t-\frac{4a_3}{t+1}a_tW\right)\nabla u.$$

Also define the function

$$Y(t):=Wu_{tt}^2+aW|\nabla u_t|^2+2a_tW\nabla u\cdot\nabla u_t,$$

and note that $Y(t)$ is continuously differentiable as an $L^1\left(\mathbb{R}^N\right)$ function by Lemma 2.4. Next, multiply $\partial_t\left(u_{tt}+u_t-\nabla\cdot\big(a\nabla u\big)\right)=0$ by $2Wu_{tt}$ to get

$$\begin{aligned}Y'(t)=&Z_1(t)+Z_2(t)-2Z_3(t)\,u_{tt}+2Z_4(t)\cdot\nabla u_t\\&+3a_tW|\nabla u_t|^2+W_t\,u_{tt}^2+\frac{8a_3}{t+1}a_tW\nabla u\cdot\nabla u_t.\end{aligned}$$

As in the proof of the weighted energy estimate Proposition 2.18, we get $Z_2(t)\le 0$. Consequently, $Z_2(t)\le 0$, assumption (B2) and $W_t\le 0$ give

$$\begin{aligned}Y'(t)\le&Z_1(t)-2Z_3(t)\,u_{tt}+2Z_4(t)\cdot\nabla u_t\\&+\frac{3a_3}{t+1}aW|\nabla u_t|^2+\frac{8a_3}{t+1}a_tW\nabla u\cdot\nabla u_t.\end{aligned} \tag{2.39}$$

To refine (2.39), apply the following inequalities, which are proved via Young's inequality:

$$2\,|Z_3(t)|\,|u_{tt}| \le \frac{t+1}{4a_3W}Z_3(t)^2 + \frac{4a_3W}{t+1}u_{tt}^2,$$
$$2\,|Z_4(t)\cdot\nabla u_t| \le \frac{t+1}{a_3aW}Z_4(t)^2 + \frac{a_3aW}{t+1}\,|\nabla u_t|^2\,.$$

The refinement is

$$Y'(t) \le Z_1(t) + \frac{t+1}{4a_3W}Z_3(t)^2 + \frac{t+1}{a_3aW}Z_4(t)^2 + \frac{4a_3}{t+1}Y(t). \tag{2.40}$$

Recall that $m \le e^m$ for all $m \in \mathbb{R}$. Thus $|\nabla W|^2 \le \frac{4\gamma W^3}{t+1}$ and $W_t^2 \le \frac{W^4}{(t+1)^2}$. Therefore, $Z_3(t)^2 \le C(a_3)\frac{aW^3|\nabla u|^2}{(t+1)^3}$ and $Z_4(t)^2 \le C(a_3,a_4)\frac{a^2W^4|\nabla u|^2}{(t+1)^4}$, by assumptions (B1) - (B3) and $0<\gamma\le\frac{1}{6a_2}$. Apply these estimates for $Z_3(t)^2$ and $Z_4(t)^2$ to (2.40) and obtain

$$Y'(t) \le Z_1(t) + C_1(a_3,a_4)\frac{aW^3\,|\nabla u|^2}{(t+1)^2} + \frac{4a_3}{t+1}Y(t). \tag{2.41}$$

For $\gamma' = 3\gamma$, define the weight $W' := \exp\left(\gamma'\frac{|x|^2}{t+1}\right)$, and notice that $W^3 = W'$. Thus (2.41) is the same as

$$(t+1)^{4a_3}\,\partial_t\left((t+1)^{-4a_3}\,Y(t)\right) \le Z_1(t) + C_1(a_3,a_4)\frac{aW'\,|\nabla u|^2}{(t+1)^2}. \tag{2.42}$$

Integrate both sides of (2.42) with respect to x, in $\mathbb{R}^N$. Now $\int_{\mathbb{R}^N} Z_1(t)dx = 0$ since u has compact support in x. Also, $\int_{\mathbb{R}^N} aW'\,|\nabla u|^2dx \le (t+1)^{a_3}\,E_{1,W}(0;u)$ by weighted energy estimate Proposition 2.18, since $0<\gamma'\le\frac{1}{2a_2}$. Therefore,

$$\partial_t\left((t+1)^{-4a_3}\int_{\mathbb{R}^N} Y(t)dx\right) \le \frac{C_1(a_3,a_4)}{(t+1)^{2+3a_3}}E_{1,W}(0;u).$$

Now integrate this inequality with respect to t, on $[0,r]$ and get

$$\int_{\mathbb{R}^N} Y(r)dx \le (r+1)^{4a_3}\,C_2(a_3,a_4)\left(E_{1,W}(0;u) + \int_{\mathbb{R}^N} Y(0)dx\right). \tag{2.43}$$

Observe that

$$\int_{\mathbb{R}^N} Y(r)dx = E_{2,W}(r;u) + \int_{\mathbb{R}^N} 2a_r W \nabla u \cdot \nabla u_r dx \tag{2.44}$$

for $r \geq 0$. Hence, we estimate the second term on the RHS of (2.44). Notice that Young's inequality and assumption (B2) give

$$|2a_r W \nabla u \cdot \nabla u_r| \leq \frac{aW}{2} |\nabla u_r|^2 + \frac{2a_r^2}{a} W |\nabla u|^2 \leq \frac{aW}{2} |\nabla u_r|^2 + \frac{2\,(a_3)^2}{(r+1)^2} aW |\nabla u|^2 .$$

Thus, by the weighted energy estimate Proposition 2.18,

$$\left| \int_{\mathbb{R}^N} 2a_r W \nabla u \cdot \nabla u_r dx \right| \leq \frac{1}{2} E_{2,W}(r;u) + \frac{2\,(a_3)^2}{(r+1)^{2-a_3}} E_{1,W}(0;u). \tag{2.45}$$

To complete the proof, apply (2.44) and (2.45) to estimate the LHS and RHS of (2.43) from below and above, respectively. We obtain

$$\frac{1}{2} E_{2,W}(r;u) \leq (r+1)^{4a_3}\, C_3(a_3,a_4) \big(E_{1,W}(0;u) + E_{2,W}(0;u) \big).$$

□

The next proposition shows that derivatives of the solution to (2.1) decay exponentially outside of a ball.

Proposition 2.20. *(Exponential decay) Let $u(x,t)$ be the solution to (2.1), and let the assumptions in subsection 2.1.1 be satisfied. For $\delta > 0$ and $\Omega(t) = \{x \in \mathbb{R}^N : |x| \geq (t+1)^{(1+\delta)/2}\}$,*

$$\| \partial_t^m \nabla^n u \|^2_{L^1(\Omega(t))} \leq C\ e^{-k(t+1)^\delta}\ \| (u_0,u_1) \|^2_{H^2 \times H^1(\mathbb{R}^N)} \tag{2.46}$$

for some $k > 0$, where $n = 0, 1$ and $m = 1-n, 2-n$. The constant C depends on $a_1, a_2, a_3, a_4, R_0,$ and δ.

Proof. Consider the notation $aW := a(x,t)\, W(x,t)$, where W is the weight from (2.33), with $\gamma = \frac{1}{6a_2}$. Then observe that

$$\| \partial_t^m \nabla^n u \|^2_{L^1(\Omega(t))} \leq \left\| \left(\sqrt{aW} \right)^{-1} \right\|^2_{L^2(\Omega(t))} \left\| \left(\sqrt{aW} \right) \partial_t^m \nabla^n u \right\|^2_{L^2(\Omega(t))} \tag{2.47}$$

by Hölder's inequality. Consequently, for $\gamma = \frac{1}{6a_2}$ in the weight function W,

$$\left\| \left(\sqrt{aW} \right) \partial_t^m \nabla^n u \right\|^2_{L^2(\Omega(t))} \leq (t+1)^{4a_3}\, C(a_3, a_4) \left(E_{1,W}(0;u) + E_{2,W}(0;u) \right) \tag{2.48}$$

by the weighted energy estimates Propositions 2.18 and 2.19. Notice that

$$\left\| \left(\sqrt{aW} \right)^{-1} \right\|^2_{L^2(\Omega(t))} \leq (t+1)^{\frac{N}{2}(1+\delta)}\, \frac{C(N)}{a_1} \int_1^\infty e^{-\gamma(t+1)^\delta r^2} r^{N-1} dr$$

by assumption (B1) and polar coordinates. Additionally,
since $r^{N-2} e^{-\frac{\gamma}{2} r^2} \leq C(\gamma, N)$ for a sufficiently large $C(\gamma, N)$ and $r \geq 1$,

$$\int_1^\infty e^{-\gamma(t+1)^\delta r^2} r^{N-1} dr \leq C(\gamma, N) \int_1^\infty e^{-\frac{\gamma}{2}(t+1)^\delta r^2} r\, dr = e^{-\frac{\gamma}{2}(t+1)^\delta} \frac{C_1(\gamma, N)}{(t+1)^\delta}.$$

Therefore,

$$\left\| \left(\sqrt{aW} \right)^{-1} \right\|^2_{L^2(\Omega(t))} \leq e^{-\frac{\gamma}{2}(t+1)^\delta}\, (t+1)^{\frac{N}{2}(1+\delta)-\delta}\, C(a_1, \gamma, N). \tag{2.49}$$

Combine (2.47) - (2.49) to get

$$\|\partial_t^m \nabla^n u\|^2_{L^1(\Omega(t))} \leq e^{-\frac{\gamma}{2}(t+1)^\delta}\, (t+1)^{\frac{N}{2}(1+\delta)-\delta+4a_3}\, C(a_1, a_3, a_4, \gamma, N) \left(E_{1,W}(0;u) + E_{2,W}(0;u) \right).$$

Note that $e^{-\frac{\gamma}{4}(t+1)^\delta}\, (t+1)^{\frac{N}{2}(1+\delta)-\delta+4a_3} \leq C(a_3, \gamma, N, \delta)$ for a sufficiently large $C(a_3, \gamma, N, \delta)$. Also, $E_{1,W}(0;u) + E_{2,W}(0;u) \leq C(a_2, \gamma, R_0)\, \|(u_0, u_1)\|^2_{H^2 \times H^1(\mathbb{R}^N)}$, where R_0 is the size of the support for the data (u_0, u_1). Therefore, to complete the proof, choose $k = \frac{\gamma}{4}$, and replace the dependence on γ with a dependence on a_2 since $\gamma = \frac{1}{6a_2}$. $\square$

2.4 The representation of the difference between solutions of (2.1) and (2.2) in terms of the fundamental solution of the parabolic problem (2.2)

The differential equation in (2.2) has a pointwise, classical fundamental solution $\Gamma(x, t; \xi, s)$ for $x, \xi \in \mathbb{R}^N$ and $0 \leq s < t$. The fundamental solution allows the transfer of decay from the solution of (2.2) to the solution of (2.1). The properties of $\Gamma(x,t;\xi,s)$ are in Friedman [5, Chapter 1]. Importantly, Friedman [5, Chapter 1, (6.12)] and [5, Chapter 1, (8.14) and Theorem 15] give the following lemma.

Lemma 2.21. *(Fundamental solution decay properties) Let $\Gamma(x,t;\xi,s)$ be the fundamental solution of (2.2). Then*

$$|\Gamma(x,t;\xi,s)| \leq C\ (t-s)^{-\frac{N}{2}}\, exp\left(-C\frac{|x-\xi|^2}{t-s}\right), \tag{i}$$

$$|\partial_{\xi_i}\Gamma(x,t;\xi,s)| \leq C\ (t-s)^{-\frac{N+1}{2}}\, exp\left(-C\frac{|x-\xi|^2}{t-s}\right). \tag{ii}$$

Definition 2.22. *We use the notations:*

$$\Gamma_x^{t,s} f := \int_{\mathbb{R}^N} \Gamma(x,t;\xi,s) f(\xi,s) d\xi,$$

$$(a\nabla_\xi\Gamma)_x^{t,s}\, g := \int_{\mathbb{R}^N} a(\xi,s)\nabla_\xi\Gamma(x,t;\xi,s)\cdot g(\xi,s)d\xi,$$

for scalar $f(\xi,s)$ and vector $g(\xi,s)$, with $f(\xi,s), |g(\xi,s)| \in L^p_\xi$ for any $1 \leq p \leq \infty$. The following lemma makes use of Lemma 2.21 to get bounds for the operators $\Gamma_x^{t,s}$ and $(a\nabla_\xi\Gamma)_x^{t,s}$.

Lemma 2.23. *(Diffusion operator estimates) Let $\Gamma(x,t;\xi,s)$ be the fundamental solution of (2.2). Then for $f(x,\cdot), |g(x,\cdot)| \in L^1(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)$ and $0 \leq s < t$, the following properties hold:*

$$\left\|\Gamma_x^{t,s} f\right\|_{L^2(\mathbb{R}^N)} \leq C\ \ \|f(x,s)\|_{L^2(\mathbb{R}^N)}, \tag{i}$$

$$\left\|\Gamma_x^{t,s} f\right\|_{L^2(\mathbb{R}^N)} \leq C\ (t-s)^{-\frac{N}{4}}\, \|f(x,s)\|_{L^1(\mathbb{R}^N)}, \tag{ii}$$

$$\left\|(a\nabla_\xi\Gamma)_x^{t,s}\, g\right\|_{L^2(\mathbb{R}^N)} \leq C(a_2)\ (t-s)^{-\frac{1}{2}}\, \|g(x,s)\|_{L^2(\mathbb{R}^N)}, \tag{iii}$$

$$\left\|(a\nabla_\xi\Gamma)_x^{t,s}\, g\right\|_{L^2(\mathbb{R}^N)} \leq C(a_2)\ (t-s)^{-\frac{N+2}{4}}\, \|g(x,s)\|_{L^1(\mathbb{R}^N)}. \tag{iv}$$

Proof. We prove (i) and (ii). By Lemma 2.21(i),

$$\left|\Gamma_x^{t,s} f\right| \leq C\ (t-s)^{-\frac{N}{2}}\, exp\left(-C\frac{|x|^2}{t-s}\right) *_x \big(|f(x,s)|\big).$$

Take the $L^2(\mathbb{R}^N)$ norm of both sides of this inequality. To get (i), apply Young's convolution inequality $\|h *_x k\|_{L^2(\mathbb{R}^N)} \leq \|h\|_{L^1(\mathbb{R}^N)} \|k\|_{L^2(\mathbb{R}^N)}$, where $h = exp\left(-C\frac{|x|^2}{t-s}\right)$ and $k = |f(x,s)|$. To get (ii), let $h = |f(x,s)|$ and $k = exp\left(-C\frac{|x|^2}{t-s}\right)$. To prove (iii) and (iv), repeat the proof of (i) and (ii), except use Lemma 2.21(ii) instead of (i). $\square$

Next, we rewrite the solution $v(x,t)$ of (2.2) as

$$v(x,t) = \Gamma_x^{t,0}\left(u + u_t\right). \tag{2.50}$$

This solution satisfies

$$v(x,t) \in \mathcal{C}\left([0,\infty); L^2(\mathbb{R}^N)\right), \tag{2.51}$$

by Lemma 2.23(i) and the continuity of $\Gamma(x,t;\xi,s)$.

The next proposition precisely determines the difference between the solutions of (2.1) and (2.2) in terms of the fundamental solution of (2.2).

Proposition 2.24. *(Integral identity) Let $u(x,t)$ be the solution to (2.1), where the assumptions in subsection 2.1.1 hold. Then for $t > 0$*

$$u(x,t) = v(x,t) - \Gamma_x^{t,t/2}u_t - \int_{t/2}^{t} \Gamma_x^{t,s} u_{ss} ds + \int_0^{t/2} \left(a\nabla_\xi \Gamma\right)_x^{t,s}\left(\nabla_\xi u_s\right) ds \tag{2.52}$$

holds in the $L^2(\mathbb{R}^N)$ sense, where $v(x,t)$ is the solution to (2.2) rewritten as in (2.50).

Proof. Let $\epsilon > 0$. Mollify $u(x,t)$ in space, i.e., use a standard mollifier $\eta_\epsilon(x) = \epsilon^{-N}\eta(\epsilon^{-1}x)$ and define $u^\epsilon(x,t) := \int_{\mathbb{R}^N} \eta_\epsilon(y)\, u(x-y,t)dy$. Note that $u^\epsilon \in C^3\left(\mathbb{R}^N \times [0,\infty)\right)$ because of the regularity (2.3) and the mollification. Moreover, $supp\,(u^\epsilon) \cap \left(\mathbb{R}^N \times [0,t]\right)$ is compact because of the finite speed of propagation Lemma 2.10. Now, define

$$f^\epsilon(x,t) := u_{tt}^\epsilon(x,t) + u_t^\epsilon(x,t) - \nabla \cdot \left(a(x,t)\nabla u^\epsilon(x,t)\right).$$

Write the above identity as $u_t^\epsilon - \nabla\cdot(a\nabla u^\epsilon) = -u_{tt}^\epsilon + f^\epsilon$ and consider this as a nonhomogeneous version of (2.2). Then by the presentation and uniqueness theorems in Friedman [5, Chapter

1, Theorems 12 and 16],

$$u^\epsilon(x,t) = \Gamma_x^{t,0} u^\epsilon - \int_0^t \Gamma_x^{t,s} \left(u_{ss}^\epsilon - f^\epsilon\right) ds. \tag{2.53}$$

Apply integration by parts in s to $\int_0^{t/2} \Gamma_x^{t,s} u_{ss}^\epsilon ds$, moving the derivative from u_{ss}^ϵ to $\Gamma_x^{t,s}$, and get

$$\int_0^{t/2} \Gamma_x^{t,s} u_{ss}^\epsilon ds = \Gamma_x^{t,t/2} u_t^\epsilon - \Gamma_x^{t,0} u_t^\epsilon - \int_0^{t/2} \int_{\mathbb{R}^N} \left(\partial_s \Gamma(x,t;\xi,s)\right) u_s^\epsilon(\xi,s) d\xi ds. \tag{2.54}$$

For the last term on the RHS of (2.54), use the fact that $\Gamma(x,t;\xi,s)$ is a classical solution to the backwards problem, i.e., use

$$\partial_s \Gamma(x,t;\xi,s) = -\nabla_\xi \cdot \left(a(\xi,s) \nabla_\xi \Gamma(x,t;\xi,s)\right),$$

where $x, \xi \in \mathbb{R}^N$ and $0 \le s < t$. Then apply the divergence theorem to the last term on the RHS of (2.54) and get

$$\int_0^{t/2} \Gamma_x^{t,s} u_{ss}^\epsilon ds = \Gamma_x^{t,t/2} u_t^\epsilon - \Gamma_x^{t,0} u_t^\epsilon - \int_0^{t/2} \left(a \nabla_\xi \Gamma\right)_x^{t,s} \left(\nabla_\xi u_s^\epsilon\right) ds.$$

Using this identity, rewrite (2.53) as

$$\begin{aligned} u^\epsilon(x,t) =& v^\epsilon(x,t) - \Gamma_x^{t,t/2} u_t^\epsilon - \int_{t/2}^t \Gamma_x^{t,s} u_{ss}^\epsilon ds \\ &+ \int_0^{t/2} \left(a \nabla_\xi \Gamma\right)_x^{t,s} \left(\nabla_\xi u_s^\epsilon\right) ds + \int_0^t \Gamma_x^{t,s} f^\epsilon ds, \end{aligned} \tag{2.55}$$

where $v^\epsilon(x,t) = \Gamma_x^{t,0}\left(u^\epsilon + u_t^\epsilon\right)$. Take $\epsilon \to 0$. Using the regularity (2.3), the first three terms on the RHS of (2.55) converge, respectively, in $L^2(\mathbb{R}^N)$ to the first three terms on the RHS of (2.52) because of the diffusion operator estimate Lemma 2.23(i). Similarly, the fourth term on the RHS of (2.55) converges in $L^2(\mathbb{R}^N)$ to the fourth term on the RHS of (2.52) because of the diffusion operator estimate Lemma 2.23(iii). Therefore, we only need $\left\| \int_0^t \Gamma_x^{t,s} f^\epsilon ds \right\|_{L^2(\mathbb{R}^N)} \to 0$ as $\epsilon \to 0$.

Note that the diffusion operator estimate Lemma 2.23(i) gives

$$\left\| \int_0^t \Gamma_x^{t,s} f^\epsilon ds \right\|_{L^2(\mathbb{R}^N)} \leq C \int_0^t \|f^\epsilon(x,s)\|_{L^2(\mathbb{R}^N)} ds. \tag{2.56}$$

By using the regularity (2.3) and the boundedness of a and ∇a, we get:

$$\|f^\epsilon(x,s)\|_{L^2(\mathbb{R}^N)} \leq \|u_{ss}(x,s)\|_{L^2(\mathbb{R}^N)} + \|u_s(x,s)\|_{L^2(\mathbb{R}^N)} + C(a)\, \|u(x,s)\|_{H_x^2} \leq M(t),$$

for $0 \leq s \leq t$, where $M(t)$ is a real-valued function of t, and

$$\left\| f^\epsilon - \big(u_{ss} + u_s - \nabla \cdot (a\nabla u)\big) \right\|_{L^2(\mathbb{R}^N)} \to 0 \quad \text{as } \epsilon \to 0.$$

By the properties of u, see Ikawa [8, Equation (1.5)], we have$\|u_{ss} + u_s - \nabla \cdot (a\nabla u)\|_{L^2(\mathbb{R}^N)}$ $= 0$, hence $\|f^\epsilon\|_{L^2(\mathbb{R}^N)} \to 0$ as $\epsilon \to 0$. Application of the dominated convergence theorem to the RHS of (2.56) completes the proof. □

2.5 The diffusion phenomenon and decay for problem (2.1)

In the proof of Theorem 2.5, the improved decay is used to extract decay from the second and third terms on the RHS of the integral identity (2.52), after using the diffusion operator estimate Lemma 2.23(i). Then the diffusion operator estimate Lemma 2.23(iv) is used to extract decay from the fourth term on the RHS of the integral identity. This comes at the price of having to estimate $\|\nabla u_s(x,s)\|_{L^1(\mathbb{R}^N)}$, which is paid by the exponential decay Proposition 2.20, followed by another application of the improved decay.

Proof of Theorem 2.5. We consider the cases when $t < 1$ and $t \geq 1$. First, assume that $t < 1$. Then by the energy inequality Lemma 2.9,

$$\|u(x,t)\|_{L^2(\mathbb{R}^N)}^2 \leq C(1)\, \|(u_0,u_1)\|_{H^2 \times H^1(\mathbb{R}^N)}^2\,, \tag{2.57}$$

and the diffusion operator estimate Lemma 2.23(i) gives

$$\|v(x,t)\|^2_{L^2(\mathbb{R}^N)} = \left\|\Gamma_x^{t,0}(u+u_t)\right\|^2_{L^2(\mathbb{R}^N)} \le C\ \ \|u_0+u_1\|^2_{L^2(\mathbb{R}^N)}.$$

Therefore, $\|u(x,t)-v(x,t)\|^2_{L^2(\mathbb{R}^N)} \le C\ \ \|(u_0,u_1)\|^2_{H^2\times H^1(\mathbb{R}^N)}$, and Theorem 2.5 is verified for $t<1$. Now, assume that $t\ge 1$. Define the functions

$$Z_1(t) := \int_0^t (s+1)^{\frac{N-3}{2}} \|u(x,s)\|^2_{L^2(\mathbb{R}^N)}\,ds,\ \ Z_2(t) := \int_0^t (s+1)^{\frac{N}{2}} \|u_s(x,s)\|^2_{L^2(\mathbb{R}^N)}\,ds,$$
$$Z_3(t) := \int_0^t (s+1)^{\frac{N+3}{2}} \|\nabla u_s(x,s)\|^2_{L^2(\mathbb{R}^N)}\,ds,\ \ Z_4(t) := \int_0^t (s+1)^{\frac{N+4}{2}} \|u_{ss}(x,s)\|^2_{L^2(\mathbb{R}^N)}\,ds.$$

Also define the function

$$Y(t) := \int_1^t (s+1)^{\frac{N-3}{2}} \|u(x,s)-v(x,s)\|^2_{L^2(\mathbb{R}^N)}\,ds,$$

which has continuous derivative $Y'(t) = (t+1)^{\frac{N-3}{2}} \|u(x,t)-v(x,t)\|^2_{L^2(\mathbb{R}^N)}\,ds$ via the regularity (2.3) and (2.51).

In the integral identity (2.52), subtract $v(x,t)$ from both sides, and then apply $\|\cdot\|^2_{L^2(\mathbb{R}^N)}$ to obtain

$$(t+1)^{\frac{5}{2}}\,Y'(t) \le C\ \ (t+1)^{\frac{N+2}{2}}\left(I_1+I_2+I_3\right), \tag{2.58}$$

where

$$I_1 = \left\|\Gamma_x^{t,t/2}u_t\right\|^2_{L^2(\mathbb{R}^N)},\quad I_2 = \left\|\int_{t/2}^t \Gamma_x^{t,s}u_{ss}ds\right\|^2_{L^2(\mathbb{R}^N)},\ \text{ and}$$
$$I_3 = \left\|\int_0^{t/2} \left(a\nabla_\xi\Gamma\right)_x^{t,s}\left(\nabla_\xi u_s\right)ds\right\|^2_{L^2(\mathbb{R}^N)}.$$

We estimate each of I_1, I_2, and I_3. For I_1, the diffusion operator estimate Lemma 2.23(i) gives $C\ I_1 \leq \|u_t(x,t/2)\|^2_{L^2(\mathbb{R}^N)}$. Thus

$$\begin{aligned} C\ (t+1)^{\frac{N+2}{2}} I_1 \leq & (t/2+1)^{\frac{N+2}{2}} \|u_t(x,t/2)\|^2_{L^2(\mathbb{R}^N)} \\ = & \int_0^{t/2} \partial_s \left((s+1)^{\frac{N+2}{2}} \|u_s(x,s)\|^2_{L^2(\mathbb{R}^N)} \right) ds + \|u_1(x)\|^2_{L^2(\mathbb{R}^N)}, \end{aligned}$$

and using $2u_s u_{ss} \leq \frac{u_s^2}{s+1} + (s+1)u_{ss}^2$ gives

$$C\ (t+1)^{\frac{N+2}{2}} I_1 \leq Z_2(t) + Z_4(t) + \|u_1(x)\|^2_{L^2(\mathbb{R}^N)}.$$

Let $\theta = \frac{N-1}{2}$. Then by the improved decay Propositions 2.13 and 2.15, respectively,

$$\begin{aligned} Z_2(t) \leq & C\ Z_1(t) + C\ \|(u_0,u_1)\|^2_{H^2\times H^1(\mathbb{R}^N)}, \\ Z_4(t) \leq & C\ Z_1(t) + C\ \|(u_0,u_1)\|^2_{H^2\times H^1(\mathbb{R}^N)}. \end{aligned}$$

Hence

$$(t+1)^{\frac{N+2}{2}} I_1 \leq C\ Z_1(t) + C\ \|(u_0,u_1)\|^2_{H^2\times H^1(\mathbb{R}^N)}, \tag{2.59}$$

where the constant C depends on a_2, a_3, a_4, and N.

Now, proceed to estimate I_2. Observe that the diffusion operator estimate Lemma 2.23(i) gives

$$I_2 \leq \left(\int_{t/2}^t \left\| \Gamma_x^{t,s} u_{ss} \right\|_{L^2(\mathbb{R}^N)} ds \right)^2 \leq C\ \left(\int_{t/2}^t \|u_{ss}(x,s)\|_{L^2(\mathbb{R}^N)} ds \right)^2,$$

and the RHS is bounded from above by $C\ (t+1)\int_{t/2}^t \|u_{ss}(x,s)\|^2_{L^2(\mathbb{R}^N)} ds$ via Hölder's inequality. Thus

$$(t+1)^{\frac{N+2}{2}} I_2 \leq C\ (t/2+1)^{\frac{N+4}{2}} \int_{t/2}^t \|u_{ss}(x,s)\|^2_{L^2(\mathbb{R}^N)} ds \leq C\ Z_4(t).$$

Therefore, as above, use the improved decay Proposition 2.15 with $\theta = \frac{N-1}{2}$ to get

$$(t+1)^{\frac{N+2}{2}} I_2 \leq C\ Z_1(t) + C\ \|(u_0,u_1)\|^2_{H^2\times H^1(\mathbb{R}^N)}, \tag{2.60}$$

where the constant C depends on a_2, a_3, a_4, and N.

Now we address I_3. Observe that the diffusion operator estimate Lemma 2.23(iv) gives

$$I_3 \leq \left(\int_0^{t/2} C(a_2)(t-s)^{-\frac{N+2}{4}} \|\nabla u_s(x,s)\|_{L^1(\mathbb{R}^N)} \, ds \right)^2$$
$$\leq C \ (t+1)^{-\frac{N+2}{2}} \left(\int_0^{t/2} C(a_2) \|\nabla u_s(x,s)\|_{L^1(\mathbb{R}^N)} \, ds \right)^2,$$

since $t \geq 1$. Thus by Hölder's inequality,

$$(t+1)^{\frac{N+2}{2}} I_3 \leq C \ \int_0^{t/2} (s+1)^{-\frac{5}{4}} \ C(a_2)^2 ds \ \int_0^{t/2} (s+1)^{\frac{5}{4}} \|\nabla u_s(x,s)\|^2_{L^1(\mathbb{R}^N)} \, ds. \tag{2.61}$$

Then the exponential decay Proposition 2.20 with $\delta = \frac{1}{2N}$ and $\Omega(s) = \{x \in \mathbb{R}^N : |x| \geq (t+1)^{(1+\delta)/2}\}$ gives

$$\|\nabla u_s(x,s)\|^2_{L^1(\Omega(s))} \leq e^{-k(s+1)^\delta} C \ \|(u_0, u_1)\|^2_{H^2 \times H^1(\mathbb{R}^N)} .$$

Combine this with the estimate

$$\|\nabla u_s(x,s)\|^2_{L^1(\Omega(s)^c)} \leq |\Omega(s)^c| \, \|\nabla u_s(x,s)\|^2_{L^2(\Omega(s)^c)} \leq C \ (s+1)^{\frac{N}{2}+\frac{1}{4}} \|\nabla u_s(x,s)\|^2_{L^2(\Omega(s)^c)} ,$$

where $\Omega(s)^c$ is the ball $\{x \in \mathbb{R}^N : |x| < (t+1)^{(1+\delta)/2}\}$ and $|\Omega(s)^c|$ is the volume of the ball, and obtain the estimate

$$\|\nabla u_s(x,s)\|^2_{L^1(\mathbb{R}^N)} \leq (s+1)^{\frac{N}{2}+\frac{1}{4}} C \ \|\nabla u_s(x,s)\|^2_{L^2(\Omega(s)^c)} + e^{-k(s+1)^\delta} C \ \|(u_0, u_1)\|^2_{H^2 \times H^1(\mathbb{R}^N)} .$$

Apply this estimate to the RHS of (2.61) and get

$$(t+1)^{\frac{N+2}{2}} I_3 \leq C \ Z_3(t) + C \ \|(u_0, u_1)\|^2_{H^2 \times H^1(\mathbb{R}^N)} .$$

Now use the improved decay Proposition 2.14 with $\theta = \frac{N-1}{2}$ to obtain

$$(t+1)^{\frac{N+2}{2}} I_3 \leq C \ Z_1(t) + C \ \|(u_0, u_1)\|^2_{H^2 \times H^1(\mathbb{R}^N)} , \tag{2.62}$$

where the constant C depends on a_1, a_2, a_3, a_4, N, and R_0. Next, apply (2.59), (2.60) and (2.62) to the RHS of (2.58) to get

$$(t+1)^{\frac{5}{2}} Y'(t) \le C\ Z_1(t) + C\ \|(u_0, u_1)\|^2_{H^2 \times H^1(\mathbb{R}^N)}, \tag{2.63}$$

where the constant C depends on a_1, a_2, a_3, a_4, N, and R_0. Then by (2.57) and $\|u\|^2_{L^2(\mathbb{R}^N)} \le C\ \|u-v\|^2_{L^2(\mathbb{R}^N)} + C\ \|v\|^2_{L^2(\mathbb{R}^N)}$,

$$\begin{aligned} Z_1(t) &= \int_0^1 (s+1)^{\frac{N-3}{2}} \|u(x,s)\|^2_{L^2(\mathbb{R}^N)}\, ds + \int_1^t (s+1)^{\frac{N-3}{2}} \|u(x,s)\|^2_{L^2(\mathbb{R}^N)}\, ds \\ &\le C\ \|(u_0,u_1)\|^2_{H^2 \times H^1(\mathbb{R}^N)} + C\ Y(t) + C\ \int_1^t (s+1)^{\frac{N-3}{2}} \|v(x,s)\|^2_{L^2(\mathbb{R}^N)}\, ds. \end{aligned}$$

Note that

$$\|v(x,s)\|^2_{L^2(\mathbb{R}^N)} \le C\ s^{-\frac{N}{2}} \|u_0+u_1\|^2_{L^1(\mathbb{R}^N)} \le C(R_0) s^{-\frac{N}{2}} \|u_0+u_1\|^2_{L^2(\mathbb{R}^N)} \tag{2.64}$$

by the diffusion operator estimate Lemma 2.23(ii). Thus

$$Z_1(t) \le C\ \|(u_0,u_1)\|^2_{H^2 \times H^1(\mathbb{R}^N)} + C\ Y(t). \tag{2.65}$$

Therefore, estimates (2.63) and (2.65) give

$$(t+1)^{\frac{5}{2}} Y'(t) \le C\ \|(u_0,u_1)\|^2_{H^2 \times H^1(\mathbb{R}^N)} + C\ Y(t). \tag{2.66}$$

Multiply both sides of (2.66) by $(t+1)^{-\frac{5}{2}}$. Then use the integrating factor $exp\left(\frac{2C}{3}(t+1)^{-3/2}\right)$ to get $Y(t) \le C\ \|(u_0,u_1)\|^2_{H^2 \times H^1(\mathbb{R}^N)}$. Therefore, the RHS of (2.66) is bounded by the initial data, giving

$$(t+1)^{\frac{N+2}{2}} \|u(x,t) - v(x,t)\|^2_{L^2(\mathbb{R}^N)} = (t+1)^{\frac{5}{2}} Y'(t) \le C\ \|(u_0,u_1)\|^2_{H^2 \times H^1(\mathbb{R}^N)},$$

completing the proof. □

Proof of Corollary 2.6. To prove (i) for $t < 1$, use (2.57). For $t \geq 1$, use the diffusion phenomenon Theorem 2.5 and (2.64).

To prove (ii), let $\theta = \frac{N}{2}$, and notice that

$$(t+1)^{\theta+1} \|\nabla u(x,t)\|^2_{L^2(\mathbb{R}^N)} - \|\nabla u_0\|^2_{L^2(\mathbb{R}^N)} = \int_0^t \partial_s \left((s+1)^{\theta+1} \|\nabla u\|^2_{L^2(\mathbb{R}^N)} \right) ds.$$

Use $2 |\nabla u \cdot \nabla u_s| \leq \frac{|\nabla u|^2}{s+1} + (s+1) |\nabla u_s|^2$ to obtain

$$\begin{aligned} &(t+1)^{\theta+1} \|\nabla u(x,t)\|^2_{L^2(\mathbb{R}^N)} - \|\nabla u_0\|^2_{L^2(\mathbb{R}^N)} \\ \leq &(\theta+2) \int_0^t (s+1)^{\theta} \|\nabla u\|^2_{L^2(\mathbb{R}^N)} ds + \int_0^t (s+1)^{\theta+2} \|\nabla u_s\|^2_{L^2(\mathbb{R}^N)} ds. \end{aligned}$$

Apply the improved decay Propositions 2.12 and 2.14, respectively, to the first and second terms on the RHS and get

$$(t+1)^{\theta+1} \|\nabla u(x,t)\|^2_{L^2(\mathbb{R}^N)} \leq C \ \|(u_0, u_1)\|^2_{H^2\times H^1(\mathbb{R}^N)} + C \int_0^t (s+1)^{\theta-1} \|u\|^2_{L^2(\mathbb{R}^N)} ds.$$

Then use part (i) of this corollary, i.e., use $\|u(x,s)\|^2_{L^2(\mathbb{R}^N)} \leq (s+1)^{-\frac{N}{2}} C \|(u_0, u_1)\|^2_{H^2\times H^1(\mathbb{R}^N)}$ to complete part (ii) of this proof.

To prove (iii), repeat the proof of (ii) with $\theta = \frac{N}{2}$, except estimate $(t+1)^{\theta+2} \|u_t(x,t)\|^2_{L^2(\mathbb{R}^N)}$ instead of $(t+1)^{\theta+1} \|\nabla u(x,t)\|^2_{L^2(\mathbb{R}^N)}$. Similarly to above,

$$\begin{aligned} &(t+1)^{\theta+2} \|u_t(x,t)\|^2_{L^2(\mathbb{R}^N)} - \|u_1\|^2_{L^2(\mathbb{R}^N)} \\ \leq &(\theta+3) \int_0^t (s+1)^{\theta+1} \|u_s\|^2_{L^2(\mathbb{R}^N)} ds + \int_0^t (s+1)^{\theta+3} \|u_{ss}\|^2_{L^2(\mathbb{R}^N)} ds. \end{aligned}$$

Apply the improved decay Propositions 2.13 and 2.15, respectively, to the first and second terms on the RHS, and then use part (i) of this corollary. $\square$

Chapter 3

The diffusion phenomenon in metric measure spaces

The material in this chapter is in preparation for submission to a peer-reviewed journal; see Taylor and Todorova [40]. The results of this chapter generalize the results of chapter 2.

Let (X, d, m) be a separable metric measure space satisfying appropriate conditions, and let $A(t)$ be a densely defined, self-adjoint operator on $L^2(X, m)$ for each $t \in \mathbb{R}$. In this chapter, we address problem (1.1) in general, which we now restate as

$$\begin{cases} u_{tt}(x,t) + u_t(x,t) + A(t)u(x,t) = 0, & x \in X,\ t > 0, \\ (u, u_t)(x, 0) = (u_0, u_1)(x), & x \in X. \end{cases} \tag{3.1}$$

Under suitable conditions, we prove that the solution to (3.1) exhibits the diffusion phenomenon (1.3), meaning that u asymptotically behaves like a solution to (1.2), which we now restate as

$$v_t(x,t) + A(t)v(x,t) = 0, \quad x \in X,\ t > 0, \tag{3.2}$$

As a corollary to the diffusion phenomenon, we obtain decay estimate for $\|u(x,t)\|_{L^2(X)}$, $\mathcal{E}(u(x,t), u(x,t))$ and $\|u_t(x,t)\|_{L^2(X)}$. Recall that $\mathcal{E}$ is the reference Dirichlet form that was introduced in assumption (D), which we will restate below.

There are several difficulties we must overcome to obtain the results in this chapter; these results are the main results of this dissertation. The intrinsic metric ρ that was introduced in section 1.3 is defined in an abstract way, making ρ hard to understand; we will address this difficulty in sections 3.1.2 and 3.7. Also, the estimate of Sturm in [37, Corollary 2.5] for the fundamental solution $p(x,t;z,s)$ of the parabolic equation (3.2) is not sufficient to guarantee $\|v(x,t)\|_{L^2(X)} \lesssim (t+1)^{-M/4}$ as $t \to \infty$ for some $M > 0$. Thus, we impose the further condition (S) in section 3.1.2. We also make substantial changes to the proofs of existence and regularity in Lions and Magenes [21, Chapter 3, Section 8] to make them work in our case; see section 3.1.3 and appendix A. Moreover, a significant amount of effort is required to demonstrate even standard properties such as the finite speed of propagation for the solution to (3.1); see section 3.2. Recall the three key tools that were introduced in chapter 2, namely the improved decay, weighted energy and fundamental solution to (3.2). We need to significantly modify these tools to make them suitable for our work in this chapter; see sections 3.3, 3.4 and 3.5, respectively.

The main results of this chapter are applicable for the following three examples, which we will make precise and fully detail in section 3.7:

E1: The space X is $\mathbb{R}^N$, and we introduce a new metric ρ that is intrinsically linked to $A(t)$, satisfying $\frac{1}{C}|x-y| \leq \rho(x,y) \leq C|x-y|^\epsilon$ locally for some $\epsilon \in (0,1)$ and $C > 0$. This metric ρ is the same as the metric in Fefferman and Phong [3], Fefferman and Sanchez-Calle [4], Jerison and Sanchez-Calle [16], and Nagel, Stein and Wainger [25]. We define the operator in (3.1) by $A(t)f := -\sum_{i=1}^N \partial_{x_i}(a_i(x,t)\partial_{x_i} f)$, giving the Dirichlet form $\mathcal{E}_t(f,g) = \int_{\mathbb{R}^N} \sum_{i=1}^N a_i(x,t)\partial_{x_i} f\, \partial_{x_i} g\, dx$. The coefficients $a_i(x,t) \geq 0$ are bounded, and the form $\mathcal{E}_t$ is only required to satisfy $\mathcal{E}_t(f,f) + \|f\|^2_{L^2(\mathbb{R}^N)} \geq \delta\|f\|^2_{H^\epsilon(\mathbb{R}^N)}$ with $\delta > 0$ and $\epsilon \in (0,1)$ the same as above, independent of t. Hence $A(t)$ is not required to be uniformly elliptic. Note that $H^\epsilon(\mathbb{R}^N)$ denotes a fractional Sobolev space $W^{\epsilon,2}(\mathbb{R}^N)$. The coefficient $a_N(x,t)$ in Example 1, given in section 7, is zero for $x = 0$.

E2: The space (X,d,m) is a (smooth) Riemannian manifold with Riemannian metric d, Riemannian volume measure m and nonnegative Ricci curvature. The self-adjoint operator $A(t)$ satisfies $0 \leq c_1\Delta_d \leq A(t) \leq c_2\Delta_d$ for $t \in \mathbb{R}$, where Δ_d is the Laplace-Beltrami operator defined on X and $c_1, c_2 > 0$.

E3: The space X is $\mathbb{R}^N$ with a weighted L^2 inner product $\langle f, g\rangle_{L^2(X)} := \langle f\sqrt{\phi}, g\sqrt{\phi}\rangle_{L^2(\mathbb{R}^N)}$, where $0 < \phi_1 \leq \phi(x) \leq \phi_2$ for constants ϕ_1 and ϕ_2. We define the operator in (3.1) by $A(t)f := -\frac{1}{\phi(x)}\sum_{i=1}^N \partial_{x_i}(\phi(x)a_i(x,t)\partial_{x_i} f)$, giving the Dirichlet form $\mathcal{E}_t(f,g) = \int_{\mathbb{R}^N}\sum_{i=1}^N a_i(x,t)\partial_{x_i} f\,\partial_{x_i} g\,\phi(x)dx$. For $1 \leq i \leq N$, the coefficients $a_i(x,t)$ are assumed to satisfy $0 < a_{i1} \leq a_i(x,t) \leq a_{i2}$ on $\mathbb{R}^N \times \mathbb{R}$, where a_{i1} and a_{i2} are constants.

3.1 Assumptions, regularity and results

3.1.1 Assumptions on the Dirichlet form $\mathcal{E}_t$

We assume that $\mathcal{E}_t$ is a strongly local, regular Dirichlet form on $L^2(X, m)$ for each $t \in \mathbb{R}$. By strongly local, we mean $\mathcal{E}_t(f, g) = 0$ when f is constant on a neighborhood of the support of g. Also, a form is regular exactly when $D(\mathcal{E}_t) \cap C_c(X)$ is: 1) dense in $D(\mathcal{E}_t)$ with respect to the norm $(\mathcal{E}_t(f,f) + \|f\|^2_{L^2(X)})^{1/2}$ and 2) dense in $C_c(X)$ with the uniform norm.

We now restate assumption (D) as (D1). We fix a strongly local, regular Dirichlet form $\mathcal{E}$ on $L^2(X, m)$, and we assume that $D(\mathcal{E}_t) = D(\mathcal{E})$ for all $t \in \mathbb{R}$. The form $\mathcal{E}$ serves as the reference form, meaning

$$c_1\mathcal{E}(f,f) \leq \mathcal{E}_t(f,f) \leq c_2\mathcal{E}(f,f) \tag{D1}$$

for all $f \in D(\mathcal{E})$ and $t \in \mathbb{R}$, where the constants c_1 and $c_2 > 0$. As was stated in section 1.3, the behavior of this reference form is linked with the behavior of the solution to (3.2) via the intrinsic pseudo metric defined in the next subsection.

Throughout this chapter, we will apply ∂_t to $\mathcal{E}_t$, and we will work with the resulting object, hence we define the form derivative

$$(\partial_t\mathcal{E}_t)(f,g) := \lim_{h\to 0}\left(\mathcal{E}_{t+h}(f,g) - \mathcal{E}_t(f,g)\right)/h,$$

and we define $(\partial_t^2\mathcal{E}_t)(f,g)$ similarly. Now since Dirichlet forms have properties such as a Cauchy-Schwarz type inequality, see (C-S) below, one might want to make the assumption that $(\partial_t\mathcal{E}_t)(f,g)$ is also a Dirichlet form, but this is restrictive since that would not allow

$(\partial_t^i \mathcal{E}_t)(f,f) < 0$ for any $f \in D(\mathcal{E})$. To avoid this restriction, we assume that $(\partial_t^i \mathcal{E}_t) \in C(\mathbb{R})$ can be written as a finite linear combination of Dirichlet forms for $i = 1, 2$, i.e., we assume

$$(\partial_t^i \mathcal{E}_t)(f,g) = \sum\nolimits_{j=1}^{J} \alpha_{i,j} \mathcal{E}_t^j(f,g) \tag{D2}$$

for $i = 1, 2$ and $J \geq 1$, where $\alpha_{i,j} \in \mathbb{R}$ and $|\alpha_{1,j}| + |\alpha_{2,j}| > 0$ for $1 \leq j \leq J$. We assume each form $\mathcal{E}_t^j$ is a strongly local, regular Dirichlet form having $D(\mathcal{E}_t^j) \subset D(\mathcal{E}_t)$. Note that $\alpha_{i,j}$ can be negative.

Remark 3.1. *In chapter 2, the Dirichlet form $\mathcal{E}_t$ is defined via $\mathcal{E}_t(f,g) := \int_{\mathbb{R}^N} a(x,t) \nabla f(x) \cdot \nabla g(x) dx$. If $a(x,t)$ were in $C^3(\mathbb{R}^N \times \mathbb{R})$, then we would have $(\partial_t^i \mathcal{E}_t)(f,g) = \mathcal{E}_t^{2i-1}(f,g)/2 - \mathcal{E}_t^{2i}(f,g)/2$ for $i = 1, 2$, where $\mathcal{E}_t^{2i-1}(f,g) := \int_{\mathbb{R}^N} \left(|\partial_t^i a(x,t)| + \partial_t^i a(x,t)\right) \nabla f(x) \cdot \nabla g(x) dx$ and $\mathcal{E}_t^{2i}(f,g) := \int_{\mathbb{R}^N} \left(|\partial_t^i a(x,t)| - \partial_t^i a(x,t)\right) \nabla f(x) \cdot \nabla g(x) dx$.*

Recall from section 1.1 that an energy measure form Γ_t is defined via $\mathcal{E}_t(f,g) = \int_X d\Gamma_t(f,g)$. In proving the finite speed of propagation and in many other places throughout this chapter, we will "apply" ∂_t to the energy measure form Γ_t, and we will estimate the resulting object $\partial_t \Gamma_t$, which we call a *form derivation*. In particular, we will frequently need local control over the total variation of $\partial_t \Gamma_t$. Let Γ_t^j by the energy measure forms which correspond to the Dirichlet forms $\mathcal{E}_t^j$ for $1 \leq j \leq J$, and for $i = 1, 2$, define the i-th *form derivation* of the energy measure form Γ_t via

$$d(\partial_t^i \Gamma_t)(f,g) := \sum\nolimits_{j=1}^{J} \alpha_{i,j} d\Gamma_t^j(f,g).$$

Consequently, these definitions "coincide" with the limit definitions for time derivatives of Γ_t; see lemma 3.8 below. Now for $i = 1, 2$, we assume

$$\sum\nolimits_{j=1}^{J} |\alpha_{i,j}| d\Gamma_t^j(f,f) \leq \frac{c_{2+i}}{(|t|+1)^i} d\Gamma_t(f,f), \tag{D3}$$

where $c_3, c_4 > 0$ are constants, $f \in D(\mathcal{E})$ and $t \in \mathbb{R}$. Note that "$\leq$" here means less than or equal to as measures on X.

Remark 3.2. *Assumption* (D3) *generalizes assumptions* (B2) *and* (B3) *from chapter 2 via remark 3.1.*

Remark 3.3. *In proving the existence, regularity and finite speed of propagation for the solution* u *to* (3.1)*, we can weaken assumption* (D3) *to* $\sum_{j=1}^{J} |\alpha_{i,j}| d\Gamma_t^j(f,f) \leq c_{2+i}\, d\Gamma_t(f,f)$.

3.1.2 Requirements for the intrinsic metric ρ

Let $\mathcal{E}$ be the reference Dirichlet form coming from the previous subsection, and let Γ be its associated energy measure form. As in section 1.1, we have $D(\mathcal{E})_{loc} = \{f \in L^2_{loc}(X,m) :$ $\Gamma(f,f)$ is a Radon measure$\}$, and as in section 1.3, the form $\mathcal{E}$ has an associated intrinsic pseudo metric ρ on X

$$\rho(x,y) = sup\{f(x) - f(y) : f \in D(\mathcal{E})_{loc} \cap C(X), d\Gamma(f,f) \leq dm(x) \text{ on } X\}.$$

In the case when $X = \mathbb{R}^N$, m is the Lebesgue measure and $d\Gamma(f,f) = a(x)|\nabla f(x)|^2\, dx$, we see that ρ is the Euclidean distance d, if $a(x) \equiv 1$. If $a(x) < 1$ in some open set U, then ρ may be interpreted as a version of d that has been "stretched" in U, i.e., $d(y,z) \leq \rho(y,z)$ for $y,z \in U$. Similarly, if $a(x) > 1$ in U, then points in U have been "compressed" together, i.e., $\rho(y,z) \leq d(y,z)$.

We assume the topology induced by ρ is equivalent to the original topology induced by d on X, guaranteeing that ρ is a metric and $x \mapsto \rho(x,y)$ is continuous on X. We assume that for every $x \in X$ and $R > 0$, the ρ-ball $B^\rho_R(x) := \{z \in X : \rho(z,x) < R\}$ is relatively compact in X. The relative compactness of the ρ-balls is equivalent to (X,ρ) being complete via Sturm [38, Theorem 2].

Recall that m is the measure given with X. Let $p(x,t;z,s)$ be the fundamental solution of (3.2), where $x,z \in X$ and $s < t$. We have the estimate from Sturm [37, Corollary 2.5]

$$\begin{aligned} &p(x,t;z,s)^2 m\left(B^\rho_{\sqrt{C_2(t-s)}}(x)\right) m\left(B^\rho_{\sqrt{C_2(t-s)}}(z)\right) \\ \leq &C_1 exp\left(-\frac{\rho(x,z)^2}{2C_2(t-s)}\right)\left(1 + \frac{\rho(x,z)^2}{C_2(t-s)}\right)^J, \end{aligned} \tag{3.3}$$

where constants $C_1, C_2 > 0$ and $J > 2$. We note however that (3.3) alone is not sufficient to guarantee L^2-decay for p. Hence, for R sufficiently large, we assume

$$\left\| \int_X m(B_R^\rho(x))^{-1}\ m(B_R^\rho(y))^{-1} exp\left(-\frac{\rho(x,y)^2}{(2+\epsilon)R^2}\right) dm(y) \right\|_{L_x^\infty(X)} \leq CR^{-M}, \tag{S}$$

where the constants ϵ, C and $M > 0$. In section 3.7, it will be shown that examples E1 - E3 satisfy (S). Note that the LHS of (S) is an upper bound for $\|p(x,t;\cdot,s)\|_{L^2(X)}^2$ if we take $R^2 \approx t - s$. We think of M as being the largest constant satisfying

$$\|v(x,t)\|_{L^2(X)} \leq C\ (t+1)^{-\frac{M}{4}}\ \|v_0\|_{L^1(X)} \tag{3.4}$$

for $t \geq 1$, where $v(x,t)$ is a solution to (3.2) with data v_0. In the special case where (3.2) is $v_t - \Delta v = 0$ with $X = \mathbb{R}^N$, we see that ρ is the Euclidean metric on $\mathbb{R}^N$, meaning (S) is satisfied with $M = N$ if m is the Lebesgue measure. Furthermore, in this case, N is the largest constant satisfying (3.4) for $t \geq 1$.

Fix $x_0 \in X$ and let $\delta, \gamma > 0$. Define the set $A_\delta(t) := \{x \in X : \rho(x,x_0) \geq (t+1)^{(1+\delta)/2}\}$ and the function $W_\gamma = exp\left(\gamma\frac{\rho(x,x_0)^2}{t+1}\right)$. If (X,ρ,m) were N-dimensional Euclidean space with the Lebesgue measure, then we would have $m(B_R^\rho(x_0)) \leq CR^N$ and $\lim_{t\to\infty} q(t)\left\|W_\gamma^{-1}\right\|_{L^1(A_\delta(t))} = 0$ for any polynomial $q(t)$. Many other triples (X,ρ,m) also satisfy these two properties. We ask that (X,ρ,m) is not too "irregular," which means we assume

$$\lim_{t\to\infty} q(t)\left\|W_\gamma^{-1}\right\|_{L^1(A_\delta(t))} = 0 \ \text{ and } \ m(B_R^\rho(x_0)) \leq CR^M \tag{W}$$

for some $x_0 \in X$, all $\delta, \gamma > 0$, all $R > 0$ sufficiently large and all polynomials $q(t)$. In section 3.7, it will be shown that examples E1 - E3 satisfy (W). We regard $m(B_R^\rho(x_0))$ as a loss in the sense that $\|f\|_{L^1(B_R^\rho(x_0))}^2 \leq m(B_R^\rho(x_0))\,\|f\|_{L^2(B_R^\rho(x_0))}^2$, but this loss is manageable since we are assuming that M in (W) is the same as in (S) above.

3.1.3 Regularity of the solution to (3.1)

Let $\mathbb{A}$ be the nonnegative definite, self-adjoint operator associated with the reference Dirichlet form $\mathcal{E}$ from subsection 3.1.1. We introduce the Hilbert spaces $H := D(\mathcal{E})$, $V := D(\mathbb{A})$ and $W := D(\mathbb{A}^{3/2})$ equipped with their norms respectively

$$\begin{aligned}&\|f\|_H^2 := \|f\|_{L^2(X)}^2 + \mathcal{E}(f,f), \quad \|f\|_V^2 := \|f\|_H^2 + \|\mathbb{A}f\|_{L^2(X)}^2\,,\\ &\|f\|_W^2 := \|f\|_V^2 + \left\|\mathbb{A}^{3/2}f\right\|_{L^2(X)}^2.\end{aligned}$$

The standard theory of Lions and Magenes [21, chapter 3, section 8] gives:

$$u(t) \in C([0,\infty); H) \quad \text{and} \quad u_t(t) \in C([0,\infty); L^2(X)).$$

However, we need:

$$u(t) \in C([0,\infty); V) \quad \text{and} \quad u_t(t) \in C([0,\infty); H).$$

Obtaining this better regularity solely from [21] requires the distributional identity

$$\langle A(t)f, g\rangle_H = \langle f, A(t)g\rangle_H, \tag{3.5}$$

where $f, g \in V$, which essentially requires the restrictive assumption that $\mathbb{A}$ and $A(t)$ commute for all t. We "recover" (3.5) via introducing the time-dependent Hilbert spaces $H(t)$ with common domain $D(\mathcal{E})$ and equivalent norms $\|f\|_{H(t)}^2 := \|f\|_{L^2(X)}^2 + \mathcal{E}_t(f,f)$. Consequently, we have

$$\langle A(t)f, g\rangle_{H(t)} = \langle f, A(t)g\rangle_{H(t)}.$$

Improving the regularity given by [21] will most likely require several strong, hard to check assumptions. For instance, we do not expect that the set of eigenfunctions of $A(t)$ remains fixed as t changes.

Our results depend on the *conclusions* of Proposition 3.4 below, meaning that we are not restricted to the particular hypotheses of Proposition 3.4; see appendix A for a proof of Proposition 3.4.

Proposition 3.4. *(Existence and regularity) Let* (D1) - (D3) *and the assumptions in appendix A.1 be satisfied. Then for any data $u_0 \in W$ and $u_1 \in V$, there exists a unique solution to* (3.1) *such that:*

$$0 = \|A(t)u(t) + u_t(t) + u_{tt}(t)\|_{L^2(X)}, \tag{i}$$

$$u_t(t) \in C([0,\infty); H) \cap L^2([0,T]; V) \quad \text{and} \quad u_{tt}(t) \in C([0,\infty); L^2(X)) \cap L^2([0,T]; H), \tag{ii}$$

$$u(t) = u_0 + \int_0^t u_s(s)ds \quad \text{and} \quad u_t(t) = u_1 + \int_0^t u_{ss}(s)ds \tag{iii}$$

for all $t \in [0,\infty)$ and all $T > 0$.

Since $u_t(t) \in L^2([0,T]; V)$ for $T > 0$, we have $u(t) \in C([0,\infty); V)$ by the first part of (iii). Also, observe that u_{tt} is an $L^2(X)$ function, not just a distribution. This fact allows us to apply the parabolic theory of Sturm [37].

3.1.4 Results for problem (3.1)

We assume that (D1) - (D3), (S) and (W) are satisfied. We also assume that the existence and regularity from Proposition 3.4 hold. Then we obtain the following Theorem and Corollary for data $u_0 \in W$ and $u_1 \in V$ with support in $B^{\rho}_{R_0}(w)$ for some $w \in X$ and $R_0 > 0$:

Theorem 3.5. *(Diffusion phenomenon) For $t \geq 0$, the solution $u(x,t)$ to (3.1) satisfies*

$$\|u(x,t) - v(x,t)\|^2_{L^2(X)} \leq C\ (t+1)^{-\frac{M}{2}}\ \|(u_0,u_1)\|^2_{V\times H}\,,$$

where $v(x,t)$ is a prescribed solution to (3.2). The constant C depends on R_0 and c_1, c_2, c_3, c_4, M coming from (D1) - (D3), (S) *and* (W).

Note that M is the same as in conditions (S) and (W). The prescribed solution $v(x,t)$ of (3.2) will be shown to have the decay $\|v(x,t)\|^2_{L^2(X)} \leq C\ (t+1)^{-\frac{M}{2}}\ \|(u_0,u_1)\|^2_{V\times H}$. Combining this property with Theorem 3.5 gives the following corollary:

Corollary 3.6. *For $t \geq 0$, the solution $u(x,t)$ to (3.1) satisfies:*

$$\|u(x,t)\|_{L^2}^2 \leq C\ (t+1)^{-\frac{M}{2}}\ \|(u_0,u_1)\|_{V\times H}^2\,, \tag{i}$$

$$\mathcal{E}(u(x,t),u(x,t)) \leq C\ \ln(t+2)\,(t+1)^{-\frac{M}{2}-1}\ \|(u_0,u_1)\|_{V\times H}^2\,, \tag{ii}$$

$$\|u_t(x,t)\|_{L^2}^2 \leq C\ \ln(t+2)\,(t+1)^{-\frac{M}{2}-2}\ \|(u_0,u_1)\|_{V\times H}^2\,, \tag{iii}$$

where the constant C depends on R_0 and c_1, c_2, c_3, c_4, M coming from (D1) - (D3)*,* (S) *and* (W)*.*

Remark 3.7. *Theorem 3.5 could potentially be strengthened. We would need an estimate for $\mathcal{E}(p,p)$, where $p = p(z) \in H$ is the fundamental solution $p(x,t;z,s)$ of* (3.2) *with x,t and s fixed. In principle, we might expect that $\mathcal{E}(p,p)$ would behave like $\|p\|_{L^2(X)}^2/(t-s)$.*

3.2 Finite speed of propagation in metric measure spaces

In this section, we prove the finite speed of propagation for u, and in section 3.4, we improve on this finite speed of propagation. If (X,ρ,m) were N-dimensional Euclidean space with the Lebesgue measure, then the cone energy $\mathbb{E}_X(t,t;u)$ defined by (3.10) would be differentiable for $t>0$, allowing us to prove the finite speed of propagation. In general though, $\mathbb{E}_X(t,t;u)$ may not be differentiable for $t>0$, but lemma 3.10 below guarantees that $\mathbb{E}_X(t,t;u)$ is absolutely continuous, which is sufficient for proving the finite speed of propagation for u.

We now give standard energy inequalities. Define the energy and total energy, respectively, as

$$E(t;f) := \frac{1}{2}\int_X |\partial_t f(x,t)|^2 dm(x) + \frac{1}{2}\mathcal{E}_t(f,f), \qquad T_E(t;f) := E(t;f) + \frac{1}{2}\int_X |f(x,t)|^2 dm(x).$$

For any $T>0$, the solution to (3.1) satisfies the following energy inequalities for $t \in [0,T]$:

$$T_E(t;u) \leq C(c_3,T)\, T_E(0;u), \tag{3.6}$$

$$E(t;\partial_t u) \leq C_2(c_3, c_4, T)\left(E(0;\partial_t u) + T_E(0;u)\right). \tag{3.7}$$

Recall the spaces H, V and W defined in subsection 3.1.3. See appendix B for the proofs of the following two lemmas:

Lemma 3.8. *(Switching) We assume that* (D1) - (D3) *hold. Let* $g(x), h(x) \in H$*, and suppose* $f(x) \in H \cap L^2(X, d\Gamma(g,g))$*. Then for* $i = 1, 2$*,*

$$\partial_t^i \left(\int_X f d\Gamma_t(g,h) \right) = \int_X f d\left(\partial_t^i \Gamma_t\right)(g,h) \in C(\mathbb{R})$$

where Γ_t *is the energy measure form associated with* $\mathcal{E}_t$*, and* $\partial_t^i \Gamma_t$ *is the i-th form derivation of* Γ_t*.*

The above lemma states that we may freely switch the time derivative and integrals involving the energy measure form Γ_t. Now we give an integration-by-parts formula:

Lemma 3.9. *(Integration-by-parts) We assume that* (D1) *holds. Let* $f(x) \in H \cap C_c(X)$*, and suppose* $g(x) \in H$ *and* $h(x) \in V$*. Then*

$$\int_X f d\Gamma_t(g,h) + \int_X g d\Gamma_t(f,h) = 2 \int_X fg(A(t)h)dx, \tag{3.8}$$

where Γ_t *is the energy measure form associated with* $\mathcal{E}_t$*, and* $A(t)$ *is the operator in* (3.1)*.*

Let $\mathcal{Q}$ be any Dirichlet form and $\mathbb{Q}$ its associated energy measure form. An important ingredient in the proofs of lemmas 3.8 and 3.9 is a "weak" integration-by-parts formula given by Fukushima, Oshima and Takeda [6, Equation 3.2.15]

$$\int_X f d\mathbb{Q}(g,h) = \mathcal{Q}(fg,h) + \mathcal{Q}(fh,g) - \mathcal{Q}(gh,f), \tag{3.9}$$

where $f \in D(\mathcal{Q}) \cap C_c(X)$ and $g, h \in D(\mathcal{Q}) \cap L^\infty(X,m)$.

Let $x_0 \in X$, $T > 0$, and choose $c^2 \leq \frac{2}{c_2}$, where c_2 comes from assumption (D1). Define the cone function $\kappa(x,t) := \max\{T - t - c\,\rho(x,x_0), 0\}$, and note that $\kappa(x,t) \in D(\mathcal{E}) \cap C_c(X)$ for each fixed t since we are assuming the balls $B_R^\rho(x)$ are relatively compact; see Sturm [37, Subsection 1.5C]. Next, define the cone energy on the set $\mathcal{U} \subset X$

$$\begin{aligned}\mathbb{E}_{\mathcal{U}}(t_1, t_2; u) :=& \frac{1}{2}\int_{\mathcal{U}} \kappa(x, t_1)\left(|u(x, t_2)|^2 + |u_t(x, t_2)|^2\right) dm(x) \\ &+ \frac{1}{4}\int_{\mathcal{U}} \kappa(x, t_1) d\Gamma_{t_2}(u(x, t_2), u(x, t_2)),\end{aligned} \tag{3.10}$$

where $t_1, t_2 \geq 0$ and u is the solution to (3.1).

Lemma 3.10. *We assume that* (D1) - (D3) *and the regularity from Proposition 3.4 hold. The cone energy* $\mathbb{E}_X(t, t; u)$ *is Lipschitz continuous in* $[0, T]$ *for any* $T > 0$.

Proof. Suppose $t_2 \geq t_1 > 0$. We have

$$\begin{aligned}&|\mathbb{E}_X(t_2, t_2; u) - \mathbb{E}_X(t_1, t_2; u)| \\ \leq&(t_2 - t_1)\left(\frac{1}{2}\int_X \left(|u(t_2)|^2 + |u_t(t_2)|^2\right) dm(x) + \frac{1}{4}\int_X d\Gamma_{t_2}(u(t_2), u(t_2))\right) \\ \leq& C(c_2)\,(t_2 - t_1) \sup_{t\in[0,T]} \left(\|u(x, t)\|_H + \|u_t(x, t)\|_{L^2}\right).\end{aligned} \tag{3.11}$$

by assumption (D1). Similarly

$$\begin{aligned}&|\mathbb{E}_X(t_1, t_2; u) - \mathbb{E}_X(t_1, t_1; u)| \\ &\leq \int_{t_1}^{t_2} |\partial_s \mathbb{E}_X(t_1, s; u)| ds \\ &\leq C(T, c_2, c_3)\,(t_2 - t_1) \sup_{t\in[0,T]} \left(\|u(x, t)\|_H + \|u_t(x, t)\|_H + \|u_{tt}(x, t)\|_{L^2}\right).\end{aligned} \tag{3.12}$$

by the switching lemma 3.8 and assumptions (D1), (D3). Therefore,

$$|\mathbb{E}_X(t_2, t_2; u) - \mathbb{E}_X(t_1, t_1; u)| \leq (t_2 - t_1)C.$$

for $0 < t_1 \leq t_2 \leq T$ via (3.11) and (3.12). Also, $\lim_{t\to 0^+} \mathbb{E}_X(t, t; u) = \mathbb{E}_X(0, 0; u)$. □

As a corollary to the preceding lemma, we see that the cone energy $\mathbb{E}_X(t, t; u)$ is absolutely continuous. Hence the classical derivative $\frac{d}{dt}\mathbb{E}_X(t, t; u)$ exists for a.e. $t \in (0, T)$ and is

Lebesgue integrable, satisfying

$$\mathbb{E}_X(t,t;u) = \int_0^t \frac{d}{ds}\mathbb{E}_X(s,s;u)ds + \mathbb{E}_X(0,0;u) \tag{AC}$$

for $t \in [0,T]$.

For $t_0 \in (0,T)$, let $B(t_0) := B^{\rho}_{(T-t_0)/c}(x_0)$, and let $S(t_0) := \{z \in X : \rho(z,x_0) = (T-t_0)/c\}$. Observe that the cone function $\kappa(x,t) = \max\{T - t - c\,\rho(x,x_0), 0\}$ is differentiable at $t = t_0$ for any fixed $x \in X \backslash S(t_0)$. Thus $\mathbb{E}_{X\backslash S(t_0)}(t,t_0;u)$ is differentiable at $t = t_0$ for any $t_0 \in (0,T)$, by dominated convergence, meaning

$$\begin{aligned}\frac{d}{dt}\mathbb{E}_{X\backslash S(t_0)}(t,t_0;u)\bigg|_{t=t_0} &= \lim_{t\to t_0^+}\frac{1}{t-t_0}\left(\mathbb{E}_{X\backslash S(t_0)}(t,t_0;u) - \mathbb{E}_{X\backslash S(t_0)}(t_0,t_0;u)\right)\\ &= \frac{d}{dt}\mathbb{E}_{B(t_0)}(t,t_0;u)\bigg|_{t=t_0}\end{aligned} \tag{3.13}$$

for any $t_0 \in (0,T)$. Also, $\mathbb{E}_{S(t_0)}(t,t_0;u) = \mathbb{E}_X(t,t_0;u) - \mathbb{E}_{X\backslash S(t_0)}(t,t_0;u)$ is differentiable at $t = t_0$ for a.e. $t_0 \in (0,T)$, since $\mathbb{E}_X(t,t_0;u)$ is differentiable for a.e. $t \in (0,T)$. Therefore,

$$\frac{d}{dt}\mathbb{E}_{S(t_0)}(t,t_0;u)\bigg|_{t=t_0} = \lim_{t\to t_0{}^+}\frac{1}{t-t_0}\left(\mathbb{E}_{S(t_0)}(t,t_0;u) - \mathbb{E}_{S(t_0)}(t_0,t_0;u)\right) = 0 \tag{3.14}$$

for a.e. $t_0 \in (0,T)$, since $\mathbb{E}_{S(t_0)}(t,t_0;u) = 0$ for $t \geq t_0$ via the fact that $\kappa(x,t) = 0$ for $(x,t) \in S(t_0) \times [t_0,\infty)$.

Let $\mathbb{Q}$ be the energy measure form associated with some Dirichlet form $\mathcal{Q}$, and suppose $f \in L^2(X,\mathbb{Q}(h,h))$ and $g \in L^2(X,\mathbb{Q}(k,k))$. The Cauchy-Schwarz inequality for energy measure forms is

$$\left|\int_X fg d\mathbb{Q}(h,k)\right|^2 \leq \int_X f^2 d\mathbb{Q}(h,h)\int_X g^2 d\mathbb{Q}(k,k); \tag{C-S}$$

see Fukushima, Oshima and Takeda [6, Lemma 5.6.1].

3.2.1 Proof of the finite speed of propagation

Proposition 3.11. *Let $u(x,t)$ be the solution to (3.1). Assume that* (D1) - (D3) *and the regularity from Proposition 3.4 hold. Then $u(x,t)$ has a finite speed of propagation with respect to the intrinsic metric ρ, namely $\mathbb{E}_X(t,t;u) \le C(T,c_3)\mathbb{E}_X(0,0;u)$ for $0 \le t \le T$.*

Proof. By (3.13) and (3.14), we have $\frac{d}{dt}\mathbb{E}_X(t,t_0;u)\big|_{t=t_0} = \frac{d}{dt}\mathbb{E}_{B(t_0)}(t,t_0;u)\big|_{t=t_0}$ for a.e. $t_0 \in (0,T)$. By the regularity of u, the switching lemma 3.8 and dominated convergence, $\lim_{t\to t_0}\frac{1}{t-t_0}\left(\mathbb{E}_X(t,t;u) - \mathbb{E}_X(t,t_0;u)\right) = \frac{d}{dt}\mathbb{E}_X(t_0,t;u)\big|_{t=t_0}$ for all $t_0 \in (0,T)$. Thus, we see $\frac{d}{dt}\mathbb{E}_X(t,t;u)\big|_{t=t_0} = \frac{d}{dt}\left(\mathbb{E}_{B(t_0)}(t,t_0;u) + \mathbb{E}_X(t_0,t;u)\right)\big|_{t=t_0}$ for a.e. $t_0 \in (0,T)$. Replace the variable t_0 with t to obtain

$$\begin{aligned}\frac{d}{dt}\mathbb{E}_X(t,t;u) = &-\frac{1}{2}\int_{B(t)} |u|^2 + |u_t|^2 dm(x) - \frac{1}{4}\int_{B(t)} d\Gamma_t(u,u) \\ &+ \int_X \kappa\left(u + u_{tt}\right)u_t dm(x) + \frac{1}{2}\int_X \kappa d\Gamma_t(u,u_t) + \frac{1}{4}\int_X \kappa d\left(\partial_t\Gamma_t\right)(u,u)\end{aligned} \tag{3.15}$$

for a.e. $t \in (0,T)$. By the integration-by-parts lemma 3.9, with $g = u_t$ and $h = u$,

$$\frac{1}{2}\int_X \kappa d\Gamma_t(u,u_t) = \int_X \kappa u_t(A(t)u)dm(x) - \frac{1}{2}\int_X u_t d\Gamma_t(\kappa,u). \tag{3.16}$$

By assumption (D3),

$$\frac{1}{4}\int_X \kappa d\left(\partial_t\Gamma_t\right)(u,u) \le \frac{c_3}{4}\int_X \kappa d\Gamma_t(u,u) \le c_3\mathbb{E}_X(t,t;u). \tag{3.17}$$

Observe $0 \le \frac{1}{4}\int_{S(t_0)} d\Gamma_{t_0}(u(t_0),u(t_0)) \le \lim_{t\to t_0^-}\frac{-1}{t-t_0}\left(\mathbb{E}_{S(t_0)}(t,t_0;u) - \mathbb{E}_{S(t_0)}(t_0,t_0;u)\right) = 0$ for a.e. $t_0 \in (0,T)$ by (3.14), so we have

$$\left|\int_{S(t)} u_t d\Gamma_t(\kappa,u)\right|^2 \le \int_{S(t)} d\Gamma_t(u,u)\int_{S(t)} |u_t|^2 d\Gamma_t(\kappa,\kappa) = 0 \tag{3.18}$$

for a.e. $t \in (0,T)$ by the Cauchy-Schwarz inequality. By the strong locality of $\mathcal{E}_t$, we have $d\Gamma_t(\kappa,\kappa) \equiv 0$ on any relatively compact open set that is disjoint from $supp(\kappa) = \overline{B(t)}$; see

[6, Corollary 3.2.1]. Also, $\overline{B(t)} = S(t) \cup B(t)$ by Sturm [38, Proposition 1]. Thus

$$\begin{aligned}\left|\int_X u_t d\Gamma_t(\kappa, u)\right| = \left|\int_{\overline{B(t)}} u_t d\Gamma_t(\kappa, u)\right| &\leq \left|\int_{B(t)} u_t d\Gamma_t(\kappa, u)\right| + \left|\int_{S(t)} u_t d\Gamma_t(\kappa, u)\right| \\ &\leq \frac{1}{2}\int_{B(t)} d\Gamma_t(u, u) + \frac{1}{2}\int_{B(t)} |u_t|^2 d\Gamma_t(\kappa, \kappa)\end{aligned} \tag{3.19}$$

for a.e. $t \in (0, T)$ via (3.18) and the Cauchy-Schwarz inequality (C-S).

The constant c in the cone function κ was chosen so that $d\Gamma_t(\kappa, \kappa) \leq c_2 d\Gamma(\kappa, \kappa) \leq c^2 c_2 dm(x) \leq 2dm(x)$; see Sturm [37, Sections 1.2B and 1.5C]. Therefore, $|u_t|^2 d\Gamma_t(\kappa, \kappa) \leq 2|u_t|^2 dm(x)$ and

$$\frac{d}{dt}\mathbb{E}_X(t, t; u) \leq -\frac{1}{2}\int_{B(t)} |u|^2 dm(x) + \int_X \kappa\,(u + u_{tt} + A(t)u)\, u_t dm(x) + c_3\mathbb{E}_X(t, t; u).$$

via applying (3.16), (3.17) and (3.19) to (3.15). Since $2uu_t \leq u^2 + u_t^2$ and $u_{tt} + A(t)u = -u_t$ in $L^2(X)$,

$$\frac{d}{dt}\mathbb{E}_X(t, t; u) \leq (1 + c_3)\, \mathbb{E}_X(t, t; u)$$

for a.e. $t \in (0, T)$. Integrate in t from $[0, r]$ and utilize the absolutely continuity (AC) of $\mathbb{E}_X(t, t; u)$. Then apply Gronwall's lemma to obtain $\mathbb{E}_X(r, r; u) \leq C(T, c_3)\mathbb{E}_X(0, 0; u)$, completing the proof. $\square$

Remark 3.12. *Since $u_t(x, t) = u_1(x) + \int_0^t u_{ss}(x, s)ds$, we see $supp(u_{tt}(x, t)) \cap B_R^\rho(y) \subseteq supp(u_t(x, t)) \cap B_R^\rho(y)$ for all $y \in X$ and $R > 0$.*

3.3 Improved decay in metric measure spaces

Recall that the energy for a function v is defined in section 2 as $E(t; v) = \frac{1}{2}\int_X |v_t|^2 dm(x) + \frac{1}{2}\mathcal{E}_t(v, v)$. The purpose of this section is to obtain the gains in the decay rates for components of the energies in terms of u. These gains in decay are expressed in a weighted average sense. The most significant change from chapter 2 is the application of the difference quotient below.

The improved decay for the energy $E(t; u)$ is proved in the following proposition.

Proposition 3.13. *Let $u(x,t)$ be the solution to (3.1). Assume that* (D1) - (D3) *and the regularity from Proposition 3.4 hold. For $r \geq 0$ and $\theta \geq 0$,*

$$\int_0^r (t+1)^\theta E(t;u)dt \leq C \left\|(u_0,u_1)\right\|^2_{H\times L^2(X)} + C\int_0^r (t+1)^{\theta-1}\left\|u\right\|^2_{L^2(X)} dt, \tag{3.20}$$

where C depends on c_2, c_3 and θ.

Proof. We begin by taking the $L^2(X)$ inner product of equation (3.1) and $2u_t$. Then apply assumption (D3) and get

$$\partial_t \left(\left\|u_t\right\|^2_{L^2(X)} + \mathcal{E}_t(u,u)\right) + 2\left\|u_t\right\|^2_{L^2(X)} = (\partial_t\mathcal{E}_t)(u,u) \leq \frac{c_3}{t+1}\mathcal{E}_t(u,u). \tag{3.21}$$

Similarly, we take the $L^2(X)$ inner product of equation (3.1) and u to obtain

$$\partial_t \left(\langle u_t\ ,\ u\rangle_{L^2(X)} + \frac{1}{2}\left\|u\right\|^2_{L^2(X)}\right) = \left\|u_t\right\|^2_{L^2(X)} - \mathcal{E}_t(u,u). \tag{3.22}$$

Next, define the continuously differentiable function

$$Y(t) := \left\|u_t\right\|^2_{L^2(X)} + \langle u_t\ ,\ u\rangle_{L^2(X)} + \frac{1}{2}\left\|u\right\|^2_{L^2(X)} + \mathcal{E}_t(u,u).$$

Then combine (3.21) with (3.22) and add $\frac{\theta}{t+1}Y(t)$ to both sides. This gives

$$\frac{\theta}{t+1}Y(t) + Y'(t) + E(t;u) \leq \frac{\theta}{t+1}Y(t) + \frac{c_3}{t+1}\mathcal{E}_t(u,u) - E(t;u). \tag{3.23}$$

Notice that $u_t u \geq -\frac{1}{2}\left(u^2+u_t^2\right)$, giving

$$0 \leq Y(t). \tag{3.24}$$

Similarly,

$$Y(t) \leq \left\|u\right\|^2_{L^2_x} + 3E(t;u), \tag{3.25}$$

since $u_t u \le \frac{1}{2}(u^2 + u_t^2)$ and $2E(t;u) = \|u_t\|^2_{L^2(X)} + \mathcal{E}_t(u,u)$. Apply (3.25) and $\mathcal{E}_t(u,u) \le 2E(t;u)$ to the RHS of (3.23) and obtain

$$\frac{\theta}{t+1}Y(t) + Y'(t) + E(t;u) \le \frac{\theta}{t+1}\|u\|^2_{L^2(X)} + \left(\frac{3\theta+2c_3}{t+1} - 1\right)E(t;u). \tag{3.26}$$

Multiply both sides of (3.26) by the integrating factor $(t+1)^\theta$ to see that

$$\begin{aligned} &\partial_t\left((t+1)^\theta Y(t)\right) + (t+1)^\theta E(t;u) \\ \le&\theta(t+1)^{\theta-1}\|u\|^2_{L^2(X)} + (t+1)^\theta\left(\frac{3\theta+2c_3}{t+1} - 1\right)E(t;u). \end{aligned} \tag{3.27}$$

Next integrate both sides of (3.27) with respect to t, from 0 to r. To complete the proof, we estimate the integrals of the first and last terms of (3.27) by the initial data. Note that (3.24) and (3.25), followed by assumption (D1) give

$$\begin{aligned} (t+1)^\theta Y(t)\big|_{t=0}^r = (r+1)^\theta Y(r) - Y(0) \ge& 0 - \left(\|u_0\|^2_{L^2(X)} + 3E(0;u)\right) \\ \ge& -C(c_2)\|(u_0,u_1)\|^2_{H\times L^2(X)}. \end{aligned}$$

Define $T_0 := \max\{0, 3\theta + 2c_3 - 1\}$. Then for all $r \ge 0$,

$$\int_0^r (t+1)^\theta\left(\frac{3\theta+2c_3}{t+1} - 1\right)E(t;u)dt \le \int_0^{T_0}(t+1)^\theta\left(\frac{3\theta+2c_3}{t+1} - 1\right)E(t;u)dt, \tag{3.28}$$

since $\frac{3\theta+2c_3}{t+1} - 1 \le 0$ for $t \ge T_0$. Apply the energy inequality (3.6) and then assumption (D1) to the RHS of (3.28), obtaining

$$\int_0^r (t+1)^\theta\left(\frac{3\theta+2c_3}{t+1} - 1\right)E(t;u)dt \le C(c_2,c_3,\theta)\|(u_0,u_1)\|^2_{H\times L^2(X)}$$

for all $r \ge 0$. Therefore, the proof of (3.20) is complete. □

The improved decay for $\|u_t\|^2_{L^2(X)}$ is proved in the following proposition.

Proposition 3.14. *Let $u(x,t)$ be the solution to (3.1). Assume that* (D1) - (D3) *and the regularity from Proposition 3.4 hold. For $r \geq 0$ and $\theta \geq 0$,*

$$\int_0^r (t+1)^{\theta+1} \|u_t\|^2_{L^2(X)}\, dt \leq C \|(u_0,u_1)\|^2_{H\times L^2(X)} + C\int_0^r (t+1)^{\theta-1} \|u\|^2_{L^2(X)}\, dt, \tag{3.29}$$

where C depends on c_2, c_3 and θ.

Proof. Add $\frac{\theta+1}{t+1}E(t;u)$ to both sides of (3.21) to obtain

$$\frac{\theta+1}{t+1}E(t;u) + \partial_t E(t;u) \leq -2\|u_t\|^2_{L^2(X)} + \frac{\theta+1}{t+1}E(t;u) + \frac{c_3}{t+1}\mathcal{E}_t(u,u).$$

Next, bound $\mathcal{E}_t(u,u)$ from above by $2E(t;u)$, and then multiply both sides of the resulting inequality by the integrating factor $(t+1)^{\theta+1}$. This gives

$$\partial_t \left((t+1)^{\theta+1}E(t;u)\right) \leq -2(t+1)^{\theta+1}\|u_t\|^2_{L^2(X)} + (\theta+1+2c_3)(t+1)^{\theta}E(t;u).$$

Next, integrate both sides of this inequality with respect to t, from 0 to r, and note that

$$(t+1)^{\theta+1}E(t;u)\Big|_{t=0}^{r} \geq -E(0;u).$$

To complete the proof of (3.29), apply the improved decay Proposition 3.13 to the term $\int_0^r (\theta+1+2c_3)(t+1)^\theta E(t;u)dt$, obtaining the last term on the RHS of (3.29). □

If $A(t)$ were time-independent, then the proofs of the two following propositions would follow directly from the proofs of Propositions 3.13 and 3.14 via applying ∂_t to (3.1). However, the time-dependence in $A(t)$ causes increased complexity in these proofs.

Remark 3.15. *In chapter 2, the third time partial derivative of u exists. We however, do not know that $\partial_t^3 u$ exists for the solution u to* (3.1)*. To bypass this deficiency, we employ difference quotients. This makes the proof of the following proposition more technically demanding than its analog in chapter 2.*

For $h > 0$, define the difference quotient operator D^h by

$$D^h f(t) := \frac{f(t+h) - f(t)}{h},$$

where $f(t)$ is a function, an operator, or a form. Let $u^h(t) := D^h u(t)$ and $A^h(t) := D^h A(t)$, and define $w(t) := u(t+h)$. From (3.1), we have

$$u^h_{tt} + u^h_t + A(t)u^h + A^h(t)w = 0. \tag{3.30}$$

Proposition 3.16. *Let $u(x,t)$ be the solution to (3.1). Assume that* (D1) - (D3) *and the regularity from Proposition 3.4 hold. For $r \geq 0$ and $\theta \geq 0$,*

$$\int_0^r (t+1)^{\theta+2} E(t; u_t) dt \leq C \left\| (u_0, u_1) \right\|^2_{V \times H} + C \int_0^r (t+1)^{\theta-1} \left\| u \right\|^2_{L^2(X)} dt, \tag{3.31}$$

where C depends on c_2, c_3, c_4 and θ.

Proof. Define the functions

$$Z_1(t) := \frac{(c_3)^2}{(t+1)^2} \mathcal{E}_t(w,w), \quad Z_2(t) := (\partial_t \mathcal{E}_t)(u^h, u^h), \quad Z_3(t) := (\partial_t^2 \mathcal{E}_t)(w, u^h),$$
$$Z_4(t) := (\partial_t \mathcal{E}_t)(w, u^h), \quad Y(t) := 2E(t; u^h) + \left\langle u^h_t\ ,\ u^h \right\rangle_{L^2(X)} + \frac{1}{2} \left\| u^h \right\|^2_{L^2(X)} + Z_1(t) + 2Z_4(t).$$

Also define the “difference” functions

$$Q_1(t) := \frac{2(c_3)^2}{(t+1)^2} \mathcal{E}_t(w, w_t - u^h), \quad Q_2(t) := \left(D^h \mathcal{E}_t\right)(w_t, u^h) - Z_2(t),$$
$$Q_3(t) := \left(D^h \partial_t \mathcal{E}_t\right)(w, u^h) - Z_3(t), \quad Q_4(t) := \left(D^h \mathcal{E}_t\right)(w, u^h) - Z_4(t),$$
$$Q(t) := Q_1(t) + 2Q_2(t) + 2Q_3(t) - Q_4(t) - 2Q_4'(t).$$

Take the $L^2(X)$ inner product of (3.30) and $2u^h_t$ to obtain

$$\partial_t \Big(2E(t; u^h) + 2Z_4(t) \Big) = -2Q_4'(t) - 2 \left\| u^h_t \right\|^2_{L^2(X)} + 3Z_2(t) + 2Q_2(t) + 2Z_3(t) + 2Q_3(t). \tag{3.32}$$

Similarly, take the $L^2(X)$ inner product of (3.30) and u^h to obtain

$$\partial_t \left(\langle u_t^h \,,\, u^h \rangle_{L^2(X)} + \frac{1}{2} \left\| u^h \right\|_{L^2(X)}^2 \right) = \left\| u_t^h \right\|_{L^2(X)}^2 - \mathcal{E}_t(u^h, u^h) - Z_4(t) - Q_4(t). \tag{3.33}$$

The left-hand sides of (3.32) and (3.33) sum to $Y'(t) - Z_1'(t)$. Thus combining (3.32) with (3.33) and adding $\frac{\theta+2}{t+1} Y(t) + Q_1(t)$ to both sides of the result gives

$$\begin{aligned} &\frac{\theta+2}{t+1} Y(t) + Y'(t) + 2E(t;u^h) \\ =&\frac{\theta+2}{t+1} Y(t) + Z_1'(t) - Q_1(t) + 3Z_2(t) + 2Z_3(t) - Z_4(t) + Q(t). \end{aligned} \tag{3.34}$$

We proceed to estimate the RHS of (3.34) from above. First, notice that

$$\begin{aligned} Z_1'(t) - Q_1(t) =& -\frac{2}{t+1} Z_1(t) + \frac{(c_3)^2}{(t+1)^2} \left((\partial_t \mathcal{E}_t)(w,w) + 2\mathcal{E}_t(w,u^h) \right) \\ \leq& \frac{c_3}{t+1} Z_1(t) + \frac{(c_3)^2}{2(t+1)} \mathcal{E}_t(u^h,u^h), \end{aligned} \tag{3.35}$$

by assumption (D3) and $2\mathcal{E}_t(w,u^h) \leq \frac{2}{t+1}\mathcal{E}_t(w,w) + \frac{t+1}{2}\mathcal{E}_t(u^h,u^h)$. Next, assumption (D3) gives

$$Z_2(t) \leq \frac{c_3}{t+1} \mathcal{E}_t(u^h,u^h). \tag{3.36}$$

For the Dirichlet forms $\mathcal{E}_t^j$ with $1 \leq j \leq J$, we have $2\left|\mathcal{E}_t^j(w,u^h)\right| \leq \epsilon\,\mathcal{E}_t^j(w,w) + \frac{1}{\epsilon}\mathcal{E}_t^j(u^h,u^h)$ by the Cauchy-Schwarz inequality (C-S) with $\epsilon > 0$. Then with $\epsilon = \frac{4c_4}{(t+1)^2}$, we have

$$2Z_3(t) \leq \frac{4(c_4)^2}{(c_3)^2(t+1)^2} Z_1(t) + \frac{1}{4}\mathcal{E}_t(u^h,u^h) \tag{3.37}$$

by assumption (D3). Similarly, with $\epsilon = \frac{2c_3}{t+1}$, we have

$$-Z_4(t) \leq Z_1(t) + \frac{1}{4}\mathcal{E}_t(u^h,u^h). \tag{3.38}$$

Utilize (3.35) - (3.38) to estimate the RHS of (3.34) from above by

$$\frac{\theta+2}{t+1} Y(t) + C\left(c_3,c_4\right) Z_1(t) + \left(\frac{C\left(c_3\right)}{t+1} + \frac{1}{2} \right) \mathcal{E}_t(u^h,u^h) + Q(t). \tag{3.39}$$

The following estimates hold:

$$Z_1(t) \le \frac{2(c_3)^2}{(t+1)^2} E(t;w), \tag{3.40}$$

$$0 \le Y(t), \tag{3.41}$$

$$Y(t) \le \left\|u^h\right\|^2_{L^2(X)} + \frac{3}{2}\left\|u^h_t\right\|^2_{L^2(X)} + 2\mathcal{E}_t(u^h,u^h) + 2Z_1(t). \tag{3.42}$$

The proof of (3.40) follows from the definitions of $Z_1(t)$ and $E(t;w)$. To show (3.41), note $Y(t) \ge \mathcal{E}_t(u^h,u^h) + Z_1(t) + 2Z_4(t)$ since $u^h_t u^h \ge -\frac{1}{2}\left((u^h)^2 + (u^h_t)^2\right)$. Similarly to (3.38), we see $-2Z_4(t) \le Z_1(t) + \mathcal{E}_t(u^h,u^h)$, showing $Y(t) \ge 0$. The proof of (3.42) is essentially identical to the proof of (3.41).

Apply (3.40) and (3.42) to bound (3.39), and hence the RHS of (3.34), from above by

$$\begin{aligned}&\frac{\theta+2}{t+1}\left\|u^h\right\|^2_{L^2(X)} + \frac{C\left(c_3,c_4,\theta\right)}{(t+1)^2}E(t;w)\\ &+\frac{1}{2}\left(\frac{C\left(c_3,\theta\right)}{t+1}+1\right)\left(\mathcal{E}_t(u^h,u^h) + \left\|u^h_t\right\|^2_{L^2(X)}\right) + Q(t).\end{aligned} \tag{3.43}$$

Replace the RHS of (3.34) with (3.43) to obtain

$$\begin{aligned}&\frac{\theta+2}{t+1}Y(t) + Y'(t) + \frac{1}{2}E(t;u^h)\\ \le&\frac{\theta+2}{t+1}\left\|u^h\right\|^2_{L^2(X)} + \frac{C\left(c_3,c_4,\theta\right)}{(t+1)^2}E(t;w) + \left(\frac{C\left(c_3,\theta\right)}{t+1} - \frac{1}{2}\right)E(t;u^h) + Q(t),\end{aligned} \tag{3.44}$$

recalling that $\mathcal{E}_t(u^h,u^h) + \left\|u^h_t\right\|^2_{L^2(X)} = 2E(t;u^h)$. Multiply both sides of (3.44) by the integrating factor $(t+1)^{\theta+2}$ and integrate in t, from 0 to r. Then let $h \to 0^+$. Observe that $Q_1(t)$ through $Q_4(t)$ approach zero uniformly on $[0,T]$ as $h \to 0^+$ because of the regularity of u and $\mathcal{E}_t$ and because of the fundamental theorem of calculus. Also, integration by parts gives $\int_0^r (t+1)^{\theta+2}Q_4'(t)dt = (t+1)^{\theta+2}Q_4(t)\big|_{t=0}^r - (\theta+2)\int_0^r (t+1)^{\theta+1}Q_4(t)dt \to 0$ as $h \to 0^+$.

Therefore, $\int_0^r (t+1)^{\theta+2} Q(t)dt \to 0$ as $h \to 0^+$. By inequalities (3.41) and (3.42), we see

$$\begin{aligned}\lim_{h\to 0^+} \int_0^r (t+1)^{\theta+2}\left(\frac{\theta+2}{t+1}Y(t)+Y'(t)\right)dt &= \lim_{h\to 0^+} \left.(t+1)^{\theta+2}Y(t)\right|_{t=0}^r \\ &\geq -C(c_2,c_3)\left\|(u_0,u_1)\right\|_{V\times H}^2.\end{aligned}$$

Define $T_0 := \max\{0, 2C(c_3,\theta)-1\}$. Then for all $r \geq 0$,

$$\begin{aligned}&\int_0^r (t+1)^{\theta+2}\left(\frac{C(c_3,\theta)}{t+1}-\frac{1}{2}\right)E(t;u_t)dt \\ &\leq \int_0^{T_0} (t+1)^{\theta+2}\left(\frac{C(a_3,\theta)}{t+1}-\frac{1}{2}\right)E(t;u_t)dt,\end{aligned} \tag{3.45}$$

since $\frac{C(a_3,\theta)}{t+1}-\frac{1}{2} \leq 0$ for $t \geq T_0$. The RHS of (3.45) is bounded above by $C(c_2,c_3,c_4,\theta)$ $\|(u_0,u_1)\|_{V\times H}^2$ via the energy inequality (3.7) followed by assumption (D1). We therefore have

$$\begin{aligned}&\int_0^r \frac{1}{2}(t+1)^{\theta+2}E(t;u_t)dt - C(c_2,c_3,c_4,\theta)\left\|(u_0,u_1)\right\|_{V\times H}^2 \\ &\leq \int_0^r (t+1)^{\theta+2}\left(\frac{\theta+2}{t+1}\|u_t\|_{L^2(X)}^2+\frac{C(c_3,c_4,\theta)}{(t+1)^2}E(t;u)\right)dt.\end{aligned}$$

To complete the proof, apply (3.20) and (3.29) from the improved decay Proposition 3.13 to the parts of the RHS involving $E(t;u)$ and $\|u_t\|_{L^2(X)}^2$, respectively. This way, we get the last term on the RHS of (3.31). □

Proposition 3.17. *Let $u(x,t)$ be the solution to (3.1). Assume that* (D1) - (D3) *and the regularity from Proposition 3.4 hold. For $r \geq 0$ and $\theta \geq 0$,*

$$\int_0^r (t+1)^{\theta+3}\|u_{tt}\|_{L^2(X)}^2 dt \leq C\left\|(u_0,u_1)\right\|_{V\times H}^2 + C\int_0^r (t+1)^{\theta-1}\|u\|_{L^2(X)}^2 dt, \tag{3.46}$$

where C depends on c_2, c_3, c_4 and θ.

Proof. Adopt the notation in the proof of Proposition 3.16, except define the functions

$$Y(t) := 2E(t;u^h)+Z_1(t)+2Z_4(t) \quad \text{and} \quad Q(t) := Q_1(t)+2Q_2(t)+2Q_3(t)-2Q_4'(t).$$

The LHS of (3.32) is equal to $Y'(t) - Z_1'(t)$. Thus adding $\frac{\theta+3}{t+1}Y(t) + Q_1(t)$ to both sides of (3.32) gives

$$\begin{aligned}&\frac{\theta+3}{t+1}Y(t) + Y'(t) + 2\left\|u_t^h\right\|^2_{L^2(X)}\\=&\frac{\theta+3}{t+1}Y(t) + Z_1'(t) - Q_1(t) + 3Z_2(t) + 2Z_3(t) + Q(t).\end{aligned} \tag{3.47}$$

Similarly to (3.37), utilizing $\epsilon = \frac{c_4}{t+1}$ and assumption (D3), we have

$$2Z_3(t) \leq \frac{1}{t+1}\left(\frac{(c_4)^2}{(c_3)^2}Z_1(t) + \mathcal{E}_t(u^h, u^h)\right). \tag{3.48}$$

As in the proof (3.41) and (3.42), we have $2|Z_4(t)| \leq Z_1(t) + \mathcal{E}_t(u^h, u^h)$. Hence the following estimates hold:

$$0 \leq Y(t), \tag{3.49}$$

$$Y(t) \leq \left\|u_t^h\right\|^2_{L^2(X)} + 2\mathcal{E}_t(u^h, u^h) + 2Z_1(t). \tag{3.50}$$

Estimate the RHS of (3.47) with (3.35), (3.36), (3.40), (3.48) and (3.50) to obtain

$$\frac{\theta+3}{t+1}Y(t) + Y'(t) + 2\left\|u_t^h\right\|^2_{L^2(X)} \leq \frac{C(c_3,\theta)}{t+1}E(t;u^h) + \frac{C(c_3,c_4,\theta)}{(t+1)^3}E(t;w) + Q(t).$$

Multiply both sides of the above inequality by the integrating factor $(t+1)^{\theta+3}$ and integrate in t, from 0 to r. Then let $h \to 0^+$, and observe that $\lim_{h\to 0^+}\int_0^r (t+1)^{\theta+3}Q(t)dt = 0$, as in the proof of Proposition 3.16. By inequalities (3.49) and (3.50), we see

$$\begin{aligned}\lim_{h\to 0^+}\int_0^r (t+1)^{\theta+3}\left(\frac{\theta+3}{t+1}Y(t) + Y'(t)\right)dt &= \lim_{h\to 0^+}(t+1)^{\theta+3}Y(t)\Big|_{t=0}^r\\&\geq -C(c_2,c_3)\left\|(u_0,u_1)\right\|^2_{V\times H}.\end{aligned}$$

We therefore have

$$\int_0^r (t+1)^{\theta+3} \|u_{tt}\|^2_{L^2(X)} \, dt - C(c_2, c_3) \|(u_0, u_1)\|^2_{V\times H}$$
$$\leq \int_0^r (t+1)^{\theta+3} \left(\frac{C(c_3,\theta)}{t+1} E(t;u_t) + \frac{C(c_3,c_4,\theta)}{(t+1)^3} E(t;u) \right) dt.$$

To complete the proof, apply the improved decay Propositions 3.13 and 3.16 to the parts of the RHS involving $E(t;u)$ and $E(t;u_t)$, respectively. This way, we get the last term on the RHS of (3.46). $\square$

3.4 Weighted energy method in metric measure spaces

The purpose of this section is to show that $\partial_t u$ and $\partial_t^2 u$ essentially decay faster than any polynomial outside of a ball; see the fast decay Proposition 3.20. This result is an improvement on the finite speed of propagation from section 3.2. The most significant changes from chapter 2 are that we must choose the weight W much more carefully and we must work with the abstract energy measure form Γ_t.

Fix some $x_0 \in X$. For $R > 0$, define $f_R(x) := (R - \rho(x,x_0))_+ \in H \cap C_c(X)$. Let $0 \leq \chi(s) \leq 1$ be a smooth cutoff function on $\mathbb{R}$ such that $\chi \equiv 1$ for $|s| \leq R-1$ and $\chi \equiv 0$ for $|s| \geq R$. For $0 < \gamma < 1$ and $r \in \mathbb{R}$, we define

$$\Phi_{\{t\}}(r) := \chi(R-r)\, exp\left(\gamma \frac{(R-r)^2}{t+1} \right).$$

We define the weight

$$W(x,t) := \Phi_{\{t\}}(f_R(x)), \tag{3.51}$$

where $R = R(T)$ is chosen large enough so that $supp(u(t)) \subseteq B^\rho_{R-1}(x_0)$ for $t \in [0,T]$. This can be done since u has a finite speed of propagation in the intrinsic metric ρ and the initial data has compact support. Observe that $W(x,t) = exp\left(\gamma \frac{\rho(x,x_0)^2}{t+1}\right)$ on $supp(u(t))$ for $t \in [0,T]$. Since $\Phi_{\{t\}}(r)$ is smooth in r and $\Phi_{\{t\}}(0) = 0$, we have $W(x,t) \in H$ is bounded by Fukushima Oshima and Takeda [6, Inequality (3.2.27)]. We also have $W(x,t) \in C_c(X)$ with support in $B^\rho_R(x_0) := \{x : \rho(x,x_0) < R\}$. Let $h(x) \in H$, and since $\mathcal{E}_t$ is a regular form, we choose a sequence $h_m(x) \in H \cap C_c(X)$ such that $h_m \to h$ in H as $m \to \infty$. We claim

$$\int_X g\, d\Gamma_t(W, h) = \int_X 2\gamma \frac{\rho(x, x_0)}{t+1} W g\, d\Gamma_t(f_R, h), \tag{3.52}$$

where $g(x) \in L^2(X)$. By the chain rule [6, Theorem 3.2.2], we have $d\Gamma_t(W, h_m) = 2\gamma \frac{\rho(x,x_0)}{t+1} W d\Gamma_t(f_R, h_m)$ and $d\Gamma_t(W, W) = (2\gamma \frac{\rho(x,x_0)}{t+1} W)^2 d\Gamma_t(f_R, f_R)$. Since $d\Gamma_t(f_R, f_R) \leq c_2 d\Gamma(f_R, f_R) \leq c_2 dm(x)$ via Sturm [37, Sections 1.2B and 1.5C], we have $g \in L^2(X, d\Gamma_t(W, W)) \cap L^2(X, d\Gamma_t(f_R, f_R))$. Thus (3.52) is valid with h_m instead of h because of [6, Lemma 5.6.1], i.e., $\left|\int_X g\, d\Gamma_t(W, h_m)\right|^2 \leq \left(\int_X g^2\, d\Gamma_t(W, W)\right)\left(\int_X d\Gamma_t(h_m, h_m)\right) < \infty$, and similarly for the RHS of (3.52). In a similar way, $\left|\int_X g\, d\Gamma_t(W, h_m - h)\right|$ and $\left|\int_X 2\gamma \frac{\rho(x,x_0)}{t+1} W g\, d\Gamma_t(f_R, h_m - h)\right|$ $\to 0$ as $m \to 0$. Therefore, we have demonstrated (3.52).

We consider the weighted energy

$$E_W(t; v) := \int_X W(x, t)|v_t|^2 dm(x) + \frac{1}{2} \int_X W(x, t) d\Gamma_t(v, v). \tag{3.53}$$

Proposition 3.18. *Let $u(x, t)$ be the solution to (3.1). Assume that* (D1) - (D3) *and the regularity from Proposition 3.4 hold. Let γ in* (3.51) *be such that $0 < \gamma \leq \frac{1}{2c_2}$. Then for $t \geq 0$*

$$E_W(t; u) \leq (t + 1)^{c_3} E_W(0; u).$$

Proof. By lemma 3.9 with $W = f$, $u_t = g$ and $u = h$,

$$\int_X W d\Gamma_t(u_t, u) + \int_X u_t d\Gamma_t(W, u) = 2\int_X W u_t(A(t)u)dx = -2\int_X W u_t(u_{tt} + u_t)dx.$$

Hence,

$$\begin{aligned} \frac{d}{dt} E_W(t; u) = \int_X W_t |u_t|^2 dx + \frac{1}{2}\int_X W_t d\Gamma_t(u, u) - 2\int_X W|u_t|^2 dx - \int_X u_t d\Gamma_t(W, u) \\ + \frac{1}{2}\int_X W d(\partial_t \Gamma_t)(u, u). \end{aligned} \tag{3.54}$$

Apply (3.52) to $-\int_X u_t\, d\Gamma_t(W,u)$, and observe

$$\begin{aligned}
&2\left|\int_X \gamma\frac{\rho(x,x_0)}{t+1}W u_t\, d\Gamma_t(f_R,u)\right| \\
&\leq \int_{supp(u)} \frac{W}{c_2}|u_t|^2 d\Gamma_t(f_R,f_R) + \int_{supp(u)} c_2\gamma^2\frac{\rho(x,x_0)^2}{(t+1)^2}W d\Gamma_t(u,u) \\
&\leq \int_X W|u_t|^2 dx - c_2\gamma\int_X W_t d\Gamma_t(u,u),
\end{aligned}$$

for $t \in [0,T]$ by the Cauchy-Schwarz inequality (C-S) and the fact $d\Gamma_t(f_R,f_R) \leq c_2 d\Gamma(f_R,f_R) \leq c_2 dm(x)$; see Sturm [37, Sections 1.2B and 1.5C]. Also, note that $W_t = -\gamma\frac{\rho(x,x_0)^2}{(t+1)^2}W$ on $supp(u)$. Since $0<\gamma\leq\frac{1}{2c_2}$ and $W_t \leq 0$ on $supp(u)$, we estimate the RHS of (3.54) from above by $\frac{1}{2}\int_X W d(\partial_t\Gamma_t)(u,u)$. We thus conclude

$$\frac{d}{dt}E_W(t;u) \leq \frac{c_3}{t+1}E_W(t;u)$$

by assumption (D3). Apply Gronwall's lemma. □

For $h>0$, recall the difference quotient operator D^h by $D^h f(t) = \frac{f(t+h)-f(t)}{h}$, where $f(t)$ is a function, an operator, or a form. Also recall that $u^h(t) = D^h u(t)$ satisfies $u^h_{tt} + u^h_t + A(t)u^h + A^h(t)w = 0$, where $w(t) = u(t+h)$ and $A^h(t) = D^h A(t)$.

Proposition 3.19. *Let $u(x,t)$ be the solution to (3.1). Assume that* (D1) *-* (D3) *and the regularity from Proposition 3.4 hold. Let γ in* (3.51) *be such that $0<\gamma\leq\frac{1}{6c_2}$. Then for $t\geq 0$*

$$E_W(t;u_t) \leq C(c_3,c_4)\ (t+1)^{10c_3+6}\left(E_W(0;u)+E_W(0;u_t)\right).$$

Proof. Choose W such that $W = exp\left(\gamma\frac{\rho(x,x_0)^2}{t+1}\right)$ on $supp(u(t))$ for $t\in[0,T+h]$. Define the functions

$$\begin{aligned}
Z^h_1(t,s) &:= \int_X W(x,t)d(\partial_s\Gamma_s)(u^h(t),w(t)), \quad Z^h_2(t,s) := \int_X W_t(x,t)d(\partial_s\Gamma_s)(u^h(t),w(t)),\\
Z^h_3(t,s) &:= \int_X W(x,t)d(\partial_s^2\Gamma_s)(u^h(t),w(t)), \quad Z^h_4(t,s) := \int_X W(x,t)d(\partial_s\Gamma_s)(u^h(t),w_t(t)),\\
Z^h_5(t,s) &:= -\int_X 2\gamma\frac{\rho(x,x_0)}{t+1}W(x,t)u^h_t(t)d(\partial_s\Gamma_s)(f_R,w(t)),
\end{aligned}$$

and note that they are continuous is t and s by lemma 3.8, the continuity of W and the regularity of u. For $i = 1, \dots, 5$, define $Z_i(t,s)$ to be the same as its corresponding $Z_i^h(t,s)$, except where w, u^h and u_t^h are replaced by u, u_t and u_{tt}, respectively. Then by the regularity and compact support of u, the boundedness of W and assumptions (D1) and (D3), we have $Z_i^h(t,s) \to Z_i(t,s)$ uniformly in h, without regard to t or s, for $i = 1, \dots, 5$. For example, with $1 \leq j \leq J$, the Cauchy-Schwarz inequality (C-S) gives

$$\begin{aligned}
&\left| \int_X W(x,t) d\Gamma_s^j(u^h(t) - u_t(t), u(t)) \right|^2 \\
\leq& \int_X W(x,t)^2 d\Gamma_s^j(u^h(t) - u_t(t), u^h(t) - u_t(t)) \int_X d\Gamma_s^j(u(t), u(t)) \\
\leq& C(W, c_2, c_3) \|u^h(t) - u_t(t)\|_H^2 \|u(t)\|_H^2,
\end{aligned}$$

and the RHS converges uniformly in h because $u^h(t) - u_t(t) = \frac{1}{h}\int_t^{t+h} u_t(s) - u_t(t) ds$ and $u_t(t)$ is continuous in H. For $i = 5$, we utilize the compact support of u and $\int_X |u_t^h(t) - u_{tt}(t)|^2 d\Gamma(f_R, f_R) \leq \|u_t^h(t) - u_{tt}(t)\|_{L^2(X)}^2$.

By lemma 3.9 with $W = f$ and $u_t^h = g$,

$$\begin{aligned}
\int_X W d\Gamma_t(u_t^h, u^h) + \int_X u_t^h d\Gamma_t(W, u^h) =& 2\int_X W u_t^h (A(t) u^h) dx \\
=& -2 \int_X W u_t^h (u_{tt}^h + u_t^h + A^h(t) w) dx,
\end{aligned}$$

where u^h and $A^h(t)w$ are the same as in (3.30). Hence as in the proof of Proposition 3.18, we have

$$\frac{d}{dt} E_W(t; u^h) \leq \frac{c_3}{t+1} E_W(t; u^h) - 2\int_X W u_t^h A^h(t) w dx. \tag{3.55}$$

By lemma 3.9 and (3.52),

$$\begin{aligned}
2\int_X W u_t^h (A^h(t) w) dx - \int_X W d(D^h \Gamma_t)(u_t^h, w) =& \int_X 2\gamma \frac{\rho(x, x_0)}{t+1} W u_t^h d(D^h \Gamma_t)(f_R, w) \\
=& -\frac{1}{h}\int_t^{t+h} Z_5^h(t,s) ds.
\end{aligned}$$

Also,

$$\int_X W d(D^h\Gamma_t)(u_t^h, w) = \frac{d}{dt}\left(\frac{1}{h}\int_t^{t+h} Z_1^h(t,s)ds\right) - \sum_{i=2}^{4}\frac{1}{h}\int_t^{t+h} Z_i^h(t,s)ds.$$

We rewrite (3.55) as

$$\frac{d}{dt}E_W(t;u^h) \le \frac{c_3}{t+1}E_W(t;u^h) + \sum_{i=2}^{5} Z_i^h(t,t) - \frac{d}{dt}Z_1^h(t,t) + Q(t), \tag{3.56}$$

where $Q(t) := \sum_{i=2}^5 \left(\frac{1}{h}\int_t^{t+h} Z_i^h(t,s)ds - Z_i^h(t,t)\right) - \frac{d}{dt}\left(\frac{1}{h}\int_t^{t+h} Z_1^h(t,s)ds - Z_1^h(t,t)\right)$.

By assumption (D3), the Cauchy-Schwarz inequality (C-S) and the inequality $-W_t = \gamma\frac{\rho(x,x_0)^2}{(t+1)^2}W \le \frac{W^2}{t+1}$ on $supp(u)$, we see

$$\begin{aligned}
|Z_1(t,t)| &\le \frac{1}{2}E_W(t;u_t) + \frac{C(c_3)}{(t+1)^2}E_W(t;u),\\
|Z_2(t,t)| &\le \frac{1}{t+1}E_W(t;u_t) + \frac{C(c_3)}{(t+1)^3}\int_X W^3 d\Gamma_t(u,u),\\
|Z_3(t,t)| &\le \frac{1}{t+1}E_W(t;u_t) + \frac{C(c_4)}{(t+1)^3}E_W(t;u), \quad |Z_4(t,t)| \le \frac{2c_3}{t+1}E_W(t;u_t).
\end{aligned}$$

Furthermore, since $d\Gamma_t(f_R,f_R) \le c_2 dm(x)$ and $\gamma \le \frac{1}{2c_2}$, we obtain

$$\begin{aligned}
|Z_5(t,t)| &\le \frac{\gamma}{t+1}\int_X W|u_{tt}|^2 d\Gamma_t(f_R,f_R) + \frac{C(c_3)}{(t+1)^2}\int_X W^2 d\Gamma_t(u,u)\\
&\le \frac{1}{t+1}E_W(t;u_t) + \frac{C(c_3)}{(t+1)^2}\int_X W^2 d\Gamma_t(u,u).
\end{aligned}$$

We observe that $E_W(t;u)$, $\int_X W^2 d\Gamma_t(u,u)$ and $\int_X W^3 d\Gamma_t(u,u)$ are bounded from above by $(t+1)^{c_3}E_W(0;u)$ via the first weighted energy inequality Proposition 3.18 because $\gamma \le \frac{1}{6c_2}$ and $W(x,t) = exp\left(\gamma\frac{\rho(x,x_0)^2}{t+1}\right)$ on $supp(u(t))$. Therefore, $\sum_{i=2}^5 |Z_i(t,t)| \le \frac{2c_3+3}{t+1}E_W(t;u_t) + C(c_3,c_4)(t+1)^{c_3-2}E_W(0;u)$, and $|Z_1(t,t)| \le \frac{1}{2}E_W(t;u_t) + C(c_3)(t+1)^{c_3-2}E_W(0;u)$.

Integrate (3.56) in t from $[0,r]$. Then take $h \to 0^+$, and notice that $\int_0^r Q(t)dt \to 0$ as $h \to 0^+$ by the uniform convergence of the Z_i^h. Hence, the above bounds for $|Z_i(t,t)|$ give

$$\frac{1}{2}E_W(r;u_r) \leq \int_0^r \frac{3c_3+3}{t+1}E_W(t;u_t)dt + C_1(c_3,c_4)(t+1)^{c_3-2}E_W(0;u) + \frac{3}{2}E_W(0;u_r).$$

Apply Gronwall's lemma to complete the proof. □

Define the set $A_\delta(t) := \{x \in X : \rho(x,x_0) \geq (t+1)^{(1+\delta)/2}\}$ for $\delta > 0$, where x_0 does not vary. Fix the constant γ such that $0 < \gamma \leq \frac{1}{6c_2}$, and define $K_\delta(t) := \int_{A_\delta(t)} exp\left(-\gamma\frac{\rho(x,x_0)^2}{t+1}\right) dm(x)$. We are now in the position to show that $\partial_t u$ and $\partial_t^2 u$ essentially decay faster than any polynomial on $A_\delta(t)$, i.e., for $i = 1,2$, we have the following estimate via Hölder's inequality:

$$\begin{aligned}\left\|\partial_t^i u\right\|^2_{L^1_x(A_\delta(t))} \leq & \left\|\left(\sqrt{W}\right)^{-1}\right\|^2_{L^2_x(A_\delta(t)\cap supp(u(t)))} \left\|\left(\sqrt{W}\right)\partial_t^i u\right\|^2_{L^2_x(A_\delta(t))} \\ \leq & K_\delta(t)\; E_W(t;\partial_t^{i-1}u),\end{aligned} \tag{3.57}$$

where W is the weight function defined in (3.51). Recall that $W = exp\left(\gamma\frac{\rho(x,x_0)^2}{t+1}\right)$ on $supp(u(t))$ for $0 \leq t \leq T$, where the choice of T may be arbitrarily large. By the weighted energy Propositions 3.18 and 3.19, we see that the weighted energy E_W can grow at a polynomial rate in t. We overcome this polynomial rate of growth via the first part of assumption (W), i.e., $\lim_{t\to\infty} q(t)\,\|W^{-1}\|_{L^1(A_\delta(t))} = 0$ where $q(t)$ is any polynomial. In particular, $K_\delta(t)$ decays faster in time than any polynomial $q(t)$. Hence by (3.57), the weighted energy Propositions 3.18 and 3.19, and assumption (W) we have proved

Proposition 3.20. *(Fast decay) Let $u(x,t)$ be the solution to (3.1). Assume that* (D1) *-* (D3), (W) *and the regularity from Proposition 3.4 hold. For $\delta > 0$, $0 < \gamma \leq \frac{1}{6c_2}$ and $t \geq 0$ large enough,*

$$\left\|\partial_t^i u\right\|^2_{L^1_x(A_\delta(t))} \leq C\;K_\delta(t)^{1/2}\;\|(u_0,u_1)\|^2_{V\times H}\,, \tag{3.58}$$

where $i = 1,2$, and the constant C depends on c_1, c_2, c_3, c_4, R_0 and δ.

Therefore, $\partial_t u$ and $\partial_t^2 u$ are essentially confined to the ball $A_\delta(t)^c$. The second part of assumption (W), namely $m(B_R^\rho(x_0)) \le CR^M$ for R sufficiently large, allows us to capitalize on this fact. We obtain the complementary estimate to the fast decay

$$\left\|\partial_t^i u\right\|^2_{L^1_x(A_\delta(t)^c)} \le |A(t)^c| \left\|\partial_t^i u\right\|^2_{L^2_x(A_\delta(t)^c)} \le C\ (t+1)^{(1+\delta)\frac{M}{2}} \left\|\partial_t^i u\right\|^2_{L^2_x(A_\delta(t)^c)} \tag{3.59}$$

for $t \ge 0$ large enough and $i = 1, 2$.

3.5 The representation of the difference between solutions of (3.1) and (3.2) in terms of the fundamental solution of the parabolic problem (3.2)

From Sturm [37], with $s \le t$, let T_t^s and S_s^t be the transition operators associated with the parabolic and coparabolic operators $\partial_t + A(t)$ and $\partial_t - A(t)$, respectively. Also, let $p(x,t;z,s)$ be the fundamental solution to problem (3.2). Note that we use a slightly different notation for p. From [37, Proposition 2.3], we have that $p(x,t;z,s)$ is the kernel of the transition operator T_t^s, and for $f(x,r) \in L^1(X) + L^\infty(X)$, with $s < t$ and $r \in \mathbb{R}$, we note that

$$T_t^s f(x,r) := T_t^s\left(f(x,r)\right) = \langle p(x,t;z,s), f(z,r)\rangle_{L^2_z(X)}. \tag{3.60}$$

By the contraction properties of T_t^s in [37, Section 1.4 C], we have

$$\|T_t^s\|_{L^2(X)\to L^2(X)} \le 1. \tag{3.61}$$

Additionally, if $t-s$ is large enough, then

$$\|T_t^s\|_{L^1(X)\to L^2(X)} \le C\ (t-s)^{-\frac{M}{4}} \tag{3.62}$$

by condition (S) and [37, (2.20.a) and Corollary 2.5]. There is a solution to (3.2) of the form

$$v(x,t) = T_t^0 u_0. \tag{3.63}$$

Proposition 3.21. *(Integral identity) Let $u(x,t)$ be the solution to (3.1), and assume that the regularity from Proposition 3.4 hold. Then for $t > 0$*

$$u = v - \int_0^t T_t^s u_{ss} ds \tag{3.64}$$

holds in the $L^2(X)$ sense, where $v(x,t)$ is the solution to (3.2) rewritten as in (3.63).

Proof. By Sturm [37, Proposition 2.3], we see $p(x,t;z,s)$ solves the coparabolic problem $A(s)^*\, v(z,s) = -\partial_s v(z,s)$ on $X \times (0,\tau)$, with $\tau < t$. Recall $A(s)$ is self-adjoint. Thus

$$\begin{aligned} w(\tau) := &\left(\langle p(x,t;z,s), u(z,s)\rangle_{L^2_z(X)}\right)\Big|_{s=0}^{\tau} - \int_0^\tau \mathcal{E}_s(p(x,t;z,s), u(z,s))ds \\ &- \int_0^\tau \langle p(x,t;z,s), u_s(z,s)\rangle_{L^2_z(X)} ds \\ =&0 \end{aligned}$$

by [37, Section 1.4 A, Remark iii]. The two integral terms in $w(\tau)$ are equal to $\int_0^\tau \langle p(x,t;z,s), u_{ss}(z,s)\rangle_{L^2_z(X)} ds$ since u is an $L^2(X)$ solution of (3.1). Hence we conveniently rewrite $w(\tau) = T_t^\tau u(x,\tau) - T_t^0 u_0 + \int_0^\tau T_t^s u_{ss} ds$ via (3.60) since $u_{tt} \in L^2(X) \subseteq L^1(X) + L^\infty(X)$.

Let $w := u(x,t) - T_t^0 u_0 + \int_0^t T_t^s u_{ss} ds$. To complete the proof, we will utilize the convenient form of $w(\tau)$ to show that $\|w\|^2_{L^2(X)} = \lim_{\tau\to t^-} \langle w(\tau), w\rangle_{L^2(X)}$, noting that the RHS is zero. Observe that $w(\tau)$ and $w \in L^2(X)$ since $\|T_t^s\|_{L^2(X)\to L^2(X)} \leq 1$ and $u, u_{tt} \in C([0,\infty); L^2(X))$. Thus we also see that $\lim_{\tau\to t^-} \int_0^\tau T_t^s u_{ss} ds = \int_0^t T_t^s u_{ss} ds$ in $L^2(X)$. Now by [37, Lemma 1.5], with $\mathscr{H} = L^2(X)$ and $\hat{S}_\tau^t = S_\tau^t$, we have $\langle T_t^\tau u(x,\tau), w\rangle_{L^2(X)} = \langle u(x,\tau), S_\tau^t w\rangle_{L^2(X)} \to \langle u(x,t), w\rangle_{L^2(X)}$ as $\tau \to t^-$, completing the proof. $\square$

3.6 The diffusion phenomenon and decay for problem (3.1)

We now prove our main theorem via combining the improved decay and weighted energy methods with the representation for u given by (3.64).

Proof of Theorem 3.5. In assumptions (S) and (W), on page 41, we take R to be $t^{1/2}C$ for some constant $C > 0$. Thus (S) and (W) are valid only if t is large enough. Hence, we assume $t \geq T_0$ for some $T_0 \geq 1$ large enough. Note that for $0 \leq t < T_0$, the energy inequality (3.6) gives

$$\|u(x,t)\|^2_{L^2(X)} \leq C(c_3, T_0)\, \|(u_0, u_1)\|^2_{H\times L^2(X)}\,. \tag{3.65}$$

Also, the transition operator estimate (3.61) gives

$$\|v(x,t)\|^2_{L^2(X)} = \|T^s_t u_0\|^2_{L^2(X)} \leq C\ \|u_0\|^2_{L^2(X)}\,.$$

Therefore, $\|u(x,t) - v(x,t)\|^2_{L^2(X)} \leq C\ \|(u_0,u_1)\|^2_{H\times L^2(X)}$, and Theorem 3.5 is verified for $t < T_0$. Now, for $t \geq T_0$, define the function

$$Y(t) := \int_{T_0}^t (s+1)^{\frac{M-3}{2}} \|u(x,s) - v(x,s)\|^2_{L^2(X)}\, ds,$$

which has continuous derivative $Y'(t) = (t+1)^{\frac{M-3}{2}} \|u(x,t) - v(x,t)\|^2_{L^2(X)}\, ds$ via the regularity of u and Sturm [37, (1.19)]. Also define the functions

$$Z_1(t) := \int_0^t (s+1)^{\frac{M-3}{2}} \|u(x,s)\|^2_{L^2(X)}\, ds \text{ and } Z_2(t) := \int_0^t (s+1)^{\frac{M+4}{2}} \|u_{ss}(x,s)\|^2_{L^2(X)}\, ds.$$

In the integral identity (3.64), subtract $v(x,t)$ from both sides, and then apply $\|\cdot\|^2_{L^2(X)}$ to obtain

$$(t+1)^{\frac{3}{2}}\, Y'(t) \leq C\ (t+1)^{\frac{M}{2}}\, (I_1 + I_2)\,,$$

where $I_1 := \left\| \int_0^{t/2} T^s_t u_{ss} ds \right\|^2_{L^2(X)}$ and $I_2 := \left\| \int_{t/2}^t T^s_t u_{ss} ds \right\|^2_{L^2(X)}$.

For I_2, the transition operator estimate (3.61) gives

$$I_2 \leq \left(\int_{t/2}^t \|T^s_t u_{ss}\|_{L^2(X)}\, ds \right)^2 \leq \left(\int_{t/2}^t \|u_{ss}(x,s)\|_{L^2(X)}\, ds \right)^2,$$

and the RHS is bounded from above by $C\ (t+1)\int_{t/2}^{t}\|u_{ss}(x,s)\|^2_{L^2(X)}\,ds$ via Hölder's inequality. Thus

$$(t+1)^{\frac{M}{2}} I_2 \le C\ (t/2+1)^{\frac{M+2}{2}}\int_{t/2}^{t}\|u_{ss}(x,s)\|^2_{L^2(X)}\,ds \le C\ Z_2(t).$$

Now we address I_1. Observe that the transition operator estimate (3.62) gives

$$\begin{aligned} I_1 &\le \left(\int_0^{t/2} C(t-s)^{-\frac{M}{4}}\,\|u_{ss}(x,s)\|_{L^1(X)}\,ds\right)^2 \\ &\le C\ (t+1)^{-\frac{M}{2}}\left(\int_0^{T_0}\|u_{ss}(x,s)\|_{L^1(X)}\,ds+\int_{T_0}^{t/2}\|u_{ss}(x,s)\|_{L^1(X)}\,ds\right)^2, \end{aligned}$$

since $t \ge T_0 \ge 1$. For $s \le T_0$, the finite speed of propagation and the second energy inequality (3.7) give $\|u_{ss}(x,s)\|_{L^1(X)} \le C(c_2,R_0,M)\,\|u_{ss}(x,s)\|_{L^2(X)} \le C\ \|(u_0,u_1)\|_{V\times H}$. By Hölder's inequality,
$\left(\int_{T_0}^{t/2}\|u_{ss}(x,s)\|_{L^1(X)}\,ds\right)^2 \le \int_{T_0}^{t/2}(s+1)^{-\frac{5}{4}}\ ds\ \int_{T_0}^{t/2}(s+1)^{\frac{5}{4}}\ \|u_{ss}(x,s)\|^2_{L^1(X)}\,ds$, and for $s \ge T_0$, the fast decay Proposition 3.20 and the complementary estimate (3.59) with $\delta=\frac{1}{2M}$ give

$$\|u_{ss}(x,s)\|^2_{L^1(X)} \le K_\delta(s)^{1/2}\,C\ \ \|(u_0,u_1)\|^2_{V\times H} + C\ (s+1)^{\frac{M}{2}+\frac{1}{4}}\,\|u_{ss}(x,s)\|^2_{L^2_x((A(s)^c))}\,.$$

Hence, since $K_\delta(s)$ decays faster than any polynomial as $s\to\infty$,

$$(t+1)^{\frac{M}{2}} I_1 \le C\ Z_2(t) + C\ \ \|(u_0,u_1)\|^2_{V\times H}\,.$$

Therefore, combining the estimates for I_1 and I_2, we see

$$(t+1)^{\frac{3}{2}}\,Y'(t) \le C\ Z_2(t) + C\ \ \|(u_0,u_1)\|^2_{V\times H}$$

The improved decay Proposition 3.17 with $\theta=\frac{M-1}{2}$ gives

$$(t+1)^{\frac{3}{2}}\,Y'(t) \le C\ Z_1(t) + C\ \ \|(u_0,u_1)\|^2_{V\times H}\,,$$

where the constant C depends on c_2, c_3, c_4, M and R_0. Then

$$\begin{aligned} Z_1(t) \leq & C\ \|(u_0,u_1)\|^2_{V\times H} + C\ Y(t) + C\ \int_{T_0}^t (s+1)^{\frac{M-3}{2}} \|v(x,s)\|^2_{L^2(X)}\, ds \\ \leq & C\ \|(u_0,u_1)\|^2_{V\times H} + C\ Y(t) \end{aligned}$$

by (3.6) and (3.62). Therefore,

$$(t+1)^{\frac{3}{2}} Y'(t) \leq C\ \|(u_0,u_1)\|^2_{V\times H} + C\ Y(t).$$

To complete the proof, apply Gronwall's lemma. □

We now prove the corollary to our main theorem.

Proof of Corollary 3.6. To prove i for $t < T_0$, apply (3.65). For $t \geq 1$, utilize the diffusion phenomenon Theorem 3.5 and (3.62) applied to (3.63).

To prove (ii), let $\theta = \frac{M}{2}$, and notice that

$$\begin{aligned} & (t+1)^{\theta+1}\mathcal{E}(u(t),u(t)) - \mathcal{E}(u_0,u_0) \\ = & \int_0^t \partial_s \left((s+1)^{\theta+1}\mathcal{E}(u(s),u(s))\right) ds \\ \leq & (\theta+2)\int_0^t (s+1)^{\theta}\mathcal{E}(u(s),u(s))ds + \int_0^t (s+1)^{\theta+2}\mathcal{E}(u_s(s),u_s(s))ds \end{aligned}$$

via $2\,|\mathcal{E}(u(s),u_s(s))| \leq \frac{\mathcal{E}(u(s),u(s))}{s+1} + (s+1)\mathcal{E}(u_s(s),u_s(s))$ Apply assumption (D1), and then the improved decay Propositions 3.13 and 3.16, respectively, to the first and second terms on the RHS and get

$$(t+1)^{\theta+1}\mathcal{E}(u(t),u(t)) \leq C\ \|(u_0,u_1)\|^2_{V\times H} + C\int_0^t (s+1)^{\theta-1}\|u(s)\|^2_{L^2(X)}\, ds.$$

Then use part (i) of this corollary to complete the proof of part (ii).

To prove (iii), repeat the proof of (ii) with $\theta = \frac{M}{2}$, except estimate $(t+1)^{\theta+2} \|u_t(t)\|^2_{L^2(X)}$ instead of $(t+1)^{\theta+1}\mathcal{E}(u(t), u(t))$. Similarly to above,

$$\begin{aligned}&(t+1)^{\theta+2} \|u_t(t)\|^2_{L^2(X)} - \|u_1\|^2_{L^2(X)} \\ \leq&(\theta+3)\int_0^t (s+1)^{\theta+1} \|u_s(s)\|^2_{L^2(X)} ds + \int_0^t (s+1)^{\theta+3} \|u_{ss}(s)\|^2_{L^2(X)} ds.\end{aligned}$$

Apply the improved decay Propositions 3.13 and 3.17, respectively, to the first and second terms on the RHS, and then use part (i) of this corollary. □

3.7 Examples

We now give the details for examples E1 - E3 described in the introduction of this paper. In our first example, we create a nonnegative definite, self-adjoint operator $A(t)f := -\sum_{i=1}^N \partial_{x_i}(a_i(x,t)\partial_{x_i} f)$ on $L^2(\mathbb{R}^N)$ with $N \geq 2$ such that $a_N(0,t) = 0$. We note that $A(t)$ is not separable, i.e., we cannot write $A(t)$ as $f(t)B$, where $f(t) \in \mathbb{R}$ and B is a time-independent operator. We show that Theorem 3.5 is applicable, with $M = N$.

In our examples, we will show that all the necessary conditions required to apply Sturm (3.3) are satisfied, including two major conditions. These two conditions are the doubling property $m(B^\rho_{2r}(x)) \leq 2^J m(B^\rho_r(x))$ and the Sobolev-type inequality $\left(\int_{B^\rho_r(x)} |f|^{\frac{2J}{J-2}} dm(y)\right)^{\frac{J-2}{J}} \leq C_S \frac{r^2}{m(B^\rho_r(x))^{2/J}} \left(\int_{B^\rho_r(x)} d\Gamma(f,f) + \int_{B^\rho_r(x)} r^{-2} f^2 dm(y)\right)$, where $x \in X$, $r > 0$, the constant $C_S \geq 0$ and $f \in D(\mathcal{E}) \cap C_c(B^\rho_r(x))$. Observe that (UP) and (SUP) in [37] are satisfied via (D1); see remarks (i) and (ii) in [37, subsection 2.2].

Example 1. Let $N \geq 2$. For $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$, define $x' := (x_2, \ldots, x_N) \in \mathbb{R}^{N-1}$. For $y \in \mathbb{R}^{N-1}$, define $0 \leq \chi(y) \leq 1$ to be a smooth cuttoff function such that $\chi(y) = 0$ for $|y| \leq 1$ and $\chi(y) = 1$ for $|y| \geq 2$. For $i = 1, \ldots, N-1$ let $a_i(x) := 1$, and let $2a_N(x) := \frac{x_1^2}{x_1^2+1} + \chi(x')^2$.

Consider the reference form

$$\mathcal{E}(f,g) := \int_{\mathbb{R}^N} \sum_{i=1}^N a_i(x) \frac{\partial f}{\partial x_i}(x) \frac{\partial g}{\partial x_i}(x) dx,$$

for $f, g \in H^1(\mathbb{R}^N)$, and note that this form is closed in $H^2(\mathbb{R}^N)$ via the Friedrichs extension method over $L^2(\mathbb{R}^N)$ since the coefficients $a_i \in C^1(\mathbb{R}^N)$. Hence $\mathcal{E}$ is a Dirichlet form. The operator $\mathbb{A}$ associated with $\mathcal{E}$ satisfies $\mathbb{A}f = -\sum_{i=1}^N \frac{\partial}{\partial x_i}\Big(a_i(x)\frac{\partial f}{\partial x_i}(x)\Big)$.

Without loss of generality, we assume $N = 2$, and now create the operator $A(t)$. For $x_1 \in \mathbb{R}$, let $0 \le \chi_1(x_1) \le 1$ be a $C^3(\mathbb{R}^N)$ cuttoff function such that $\chi_1(x_1) = 0$ for $|x_1| \le 1$ and $\chi_1(x) = 1$ for $|x_1| \ge 2$. Then define $k(x_1, t) := \frac{t^2}{t^2+1}\chi_1(x_1)$, $B(t) := (1 + k(x_1, t))\mathbb{A}$, $C(t) := -(\frac{\partial k}{\partial x_1}(x_1, t))\frac{\partial}{\partial x_1}$ and $A(t) := B(t) + C(t)$.

We note that $\mathbb{A}$ and $A(t)$ are uniformly elliptic away from $x = 0$. Hence, one may expect that the intrinsic metric ρ behaves like the Euclidean metric at large scales. Similarly, one may expect that v, the solution to (3.2), behaves as if $A(t)$ were $-\Delta$. Thus we expect M in Theorem 3.5 and Corollary 3.6 to be exactly N.

We now verify that the conditions required to apply Theorem 3.5 are met, and then we show $M = N$. Clearly $\mathcal{E}_t$ satisfies (D1), and it also satisfies (D2), (D3) because we can separate $\frac{\partial^i k}{\partial t^i}(x_1, t)$ into its positive and negative parts to create new Dirichlet forms. Also, $\|(\partial_t^i A(t))f\|^2_{L^2(\mathbb{R}^N)} \le C(i)(\|\mathbb{A}f\|^2_{L^2(\mathbb{R}^N)} + \mathcal{E}(f, f))$ for $i = 0, 1, 2$, so assumption (A1) is satisfied. Observe that $\|A(t)f\|^2_{L^2(\mathbb{R}^N)} \ge \frac{1}{2}\|B(t)f\|^2_{L^2(\mathbb{R}^N)} - \|C(t)f\|^2_{L^2(\mathbb{R}^N)} \ge \frac{1}{2}\|\mathbb{A}f\|^2_{L^2(\mathbb{R}^N)} - C_1\mathcal{E}(f, f))$, meaning (A2) is satisfied. We note that $A(t)$ satisfies (A3).

The forms $\mathcal{E}$ and $\mathcal{E}_t$ are closed in $H^2(\mathbb{R}^N)$ via the Friedrichs extension method over $L^2(\mathbb{R}^N)$, and $C_c^\infty(\mathbb{R}^N)$ is dense in $H^2(\mathbb{R}^N)$ endowed with the H norm. Hence these forms have the same closure over $C_c^\infty(\mathbb{R}^N)$, giving us that these forms are strongly local and regular. Similarly, $C_c^\infty(\mathbb{R}^N)$ is dense in $D(\mathbb{A}) = V$. Furthermore, $C_c^\infty(\mathbb{R}^N) \subset W$, meaning the containments $W \subseteq V \subseteq H \subseteq L^2(X)$ are dense. Also $A(0) = \mathbb{A}$.

Recall that we are assuming $N = 2$ without loss of generality. As in Hörmander [7, (1.6)], we write $\mathbb{A} = X_1^2 + X_2^2 + X_3^2 + X_0$, where $X_1 = \frac{\partial}{\partial x_1}$, $X_2 = \frac{x_1}{\sqrt{2}(x_1^2+1)^{1/2}}\frac{\partial}{\partial x_2}$, $X_3 = \frac{\chi(x_2)}{\sqrt{2}}\frac{\partial}{\partial x_2}$, and $X_0 = -\frac{1}{2}\chi(x_2)\chi'(x_2)\frac{\partial}{\partial x_2}$ are C^∞ vector fields. Let $|||f|||^2 := \sum_{i=1}^3 \|X_i f\|^2_{L^2(\mathbb{R}^2)} + \|f\|^2_{L^2(\mathbb{R}^2)}$ and $|||f|||' := \sup_{g \in C_c^\infty(\mathbb{R}^2)} |\int_{\mathbb{R}^2} f(x)g(x)dx|/|||g|||$. Then since X_1 and the commutator $[X_1, X_2]$ span the tangent space at each $x \in \mathbb{R}^2$, we apply [7, (3.4)] with $K := \overline{B_4(0)} \subseteq \mathbb{R}^2$, i.e., we apply

$$\|f\|_{H^\epsilon(\mathbb{R}^2)} \le C(|||f||| + |||X_0 f|||') \tag{3.66}$$

for all $f \in C_c^\infty(K)$ and some constants C and $\epsilon > 0$. Notice that integration-by-parts followed by Hölder's inequality gives

$$\begin{aligned}2\left|\int_{\mathbb{R}^2}(X_0 f(x))g(x)dx\right| = &\left|\int_{\mathbb{R}^2} f(x)\,\frac{\partial}{\partial x_2}(\chi(x_2)\chi'(x_2)g(x))dx\right| \\ \leq &C_1\,\|f\|_{L^2(\mathbb{R}^2)}\left(\|g\|_{L^2(\mathbb{R}^2)} + \|X_3 g\|_{L^2(\mathbb{R}^2)}\right)\end{aligned}$$

for $g \in C_c^\infty(\mathbb{R}^2)$. Hence if $f \in C_c^\infty(K)$, then $|||X_0 f|||' \leq C_2\,\|f\|_{L^2}$, meaning $\|f\|_{H^\epsilon(\mathbb{R}^2)} \leq C_3\,|||f|||$ by (3.66). Therefore if $h \in C_c^\infty(\mathbb{R}^2)$, then $\|h\|_{H^\epsilon(\mathbb{R}^2)} \leq C_4\,|||h||| = C_4(\mathcal{E}(h,h) + \|h\|^2_{L^2(\mathbb{R}^2)})^{1/2}$ since A is uniformly elliptic on $\mathbb{R}^2 \setminus B_3(0)$ and $B_3(0) \subset\subset K$.

Consequently, the intrinsic metric ρ is the same as the metric in Fefferman and Phong [3], Fefferman and Sanchez-Calle [4], Jerison and and Sanchez-Calle [16], and Nagel, Stein and Wainger [25]. Thus we have the local estimate

$$\frac{1}{C}|x-y| \leq \rho(x,y) \leq C|x-y|^\epsilon, \tag{3.67}$$

so ρ generates a topology on $\mathbb{R}^2$ that is equivalent to the Euclidean topology, and (X,ρ) is complete, meaning $B_R^\rho(x)$ is relatively compact for each $x \in \mathbb{R}^2$ and $R > 0$. Furthermore, the intrinsic metric ρ satisfies a doubling property and a Sobolev-type inequality; see Biroli and Mosco [1, Page 133], and see the references therein.

Observe that $A \leq -\Delta$ on $L^2(\mathbb{R}^N)$, so

$$|x-y| \leq \rho(x,y) \tag{3.68}$$

for all $x, y \in \mathbb{R}^2$. Thus $B_R^\rho(x) \subseteq B_R(x)$ for all $x \in \mathbb{R}^2$ and $R > 0$, meaning the second part of (W) is satisfied with $M = N$ $(= 2)$. From (3.67), we see $\rho : \mathbb{R}^2 \times \mathbb{R}^2 \to [0,\infty)$ is continuous. Hence $\rho(a,b) \leq R_1$ for some constant $R_1 > 0$ and all $a, b \in \overline{B_3(0)}$. Let $L(x,y)$ be the Euclidean line from x to y. Then since $-\frac{1}{2}\Delta \leq A$ for $x \in \mathbb{R}^2 \setminus B_3(0)$, we have $\rho(x,y) \leq \sqrt{2}|x-y|$ if $L(x,y) \cap \overline{B_3(0)}$ is empty. Suppose $z_1, z_2 \in L(x,y) \cap \partial B_3(0)$, where $x, y \in \mathbb{R}^2$. Then $\rho(x,y) \leq \rho(x,z_1) + \rho(z_1,z_2) + \rho(z_2,y) \leq \sqrt{2}|x-y| + R_1$. Thus

$$B_R(x) \subseteq B^\rho_{\sqrt{2}R+R_1}(x) \tag{3.69}$$

for all $x \in \mathbb{R}^2$ and $R > 0$. Therefore, (S) and the first part of (W) are satisfied with $M = N$ $(= 2)$ via (3.68) and (3.69).

In our second example, we construct an operator on $L^2(X)$, where X is the cylinder $\mathbb{R}^N \times S^1$. We show that Theorem 3.5 is applicable, with $M = N$.

Example 2. Let X be the cylinder $\mathbb{R}^N \times S^1$, which is a Ricci-flat Riemannian manifold since the sectional curvatures are 0. We assume the self-adjoint operator $A(t)$ satisfies $0 \leq c_1\Delta_d \leq A(t) \leq c_2\Delta_d$ on $L^2(X)$ for $t \in \mathbb{R}$, where Δ_d is the Laplace-Beltrami operator defined on X. Hence we take the reference form to be

$$\mathcal{E}(f,g) := \int_X \nabla_d f(x)\, \nabla_d g(x) dm(x),$$

where ∇_d is the gradient on X. Therefore, the intrinsic metric ρ is exactly d, meaning that M in Theorem 3.5 and Corollary 3.6 is exactly N, assuming $A(t)$ satisfies the necessary conditions. Notice that constructing such an $A(t)$ is not difficult, e.g., we could have $A(t)f := -\nabla_d \cdot ((t^2 + k(\theta))/(t^2 + 2k(\theta))\nabla_d f(x))$, where $k(\theta) > 0$ is a smooth function for $\theta \in S^1$.

Since the Ricci curvature of X is nonnegative, we have that the intrinsic metric ρ satisfies a doubling property and a Sobolev-type inequality; see Saloff-Coste [35, Inequality (14) and Theorem 3.1].

In our third example, we construct an operator on $L^2(X) = \langle f\sqrt{\phi}, g\sqrt{\phi}\rangle_{L^2(\mathbb{R}^N)}$. We show that Theorem 3.5 is applicable, with $M = N$.

Example 3. Let $\phi \in C^3(\mathbb{R}^N)$ have bounded derivatives and satisfy $0 < a_1 \leq \phi(x) \leq a_2$ for $x \in \mathbb{R}^N$, where a_1, a_2 are constants. Consider the weighted L^2-space on $\mathbb{R}^N$ with inner product $\langle f, g\rangle_{L^2(X)} = \langle f\sqrt{\phi}, g\sqrt{\phi}\rangle_{L^2(\mathbb{R}^N)}$. We define the operator in (3.1) by $A(t)f := -\frac{1}{\phi(x)}\sum_{i=1}^N \partial_{x_i}(\phi(x)a_i(x,t)\partial_{x_i} f)$, giving the Dirichlet form $\mathcal{E}_t(f,g) = \int_{\mathbb{R}^N}\sum_{i=1}^N a_i(x,t)\partial_{x_i} f\, \partial_{x_i} g\, \phi(x)dx$. For $1 \leq i \leq N$, the coefficients $a_i(x,t) \in C^3(\mathbb{R}^N \times \mathbb{R})$ are assumed to have bounded derivatives and satisfy $0 < a_{i1} \leq a_i(x,t) \leq a_{i2}$ on $\mathbb{R}^N \times \mathbb{R}$, where a_{i1} and a_{i2} are constants. Since ϕ and a_i are bounded above and below by positive constants, we see that the intrinsic metric ρ is equivalent to the Euclidean metric d on

$\mathbb{R}^N$. Therefore, M in Theorem 3.5 and Corollary 3.6 is exactly N, if for instance the time derivatives of a_i decay sufficiently over time. We also note that the intrinsic metric ρ satisfies a doubling property and a Sobolev-type inequality; cf. Biroli and Mosco [1, Pages 132-133], and see the references therein.

Remark 3.22. *In example 3, with more careful analysis, it may be possible to consider a more general weight $\phi(x)$ belonging to the Muckenhoupt class A_2.*

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Appendices

A Existence, uniqueness and regularity for the solution to (3.1)

Appendix A is devoted to proving Proposition 3.4, conveniently restated below; this shows existence, uniqueness and regularity for the solution u to (3.1).

Let $A(t)$ and $\mathbb{A}$ be the operators associated with the Dirichlet forms $\mathcal{E}_t$ and $\mathcal{E}$, respectively. As in subsection 3.1.3, we have the Hilbert spaces $H = D(\mathcal{E})$, $V = D(\mathbb{A})$ and $W = D(\mathbb{A}^{3/2})$ equipped with their norms respectively

$$\begin{aligned}
&\|f\|_H^2 = \|f\|_{L^2(X)}^2 + \mathcal{E}(f,f), \quad \|f\|_V^2 = \|f\|_H^2 + \|\mathbb{A}f\|_{L^2(X)}^2, \\
&\|f\|_W^2 = \|f\|_V^2 + \left\|\mathbb{A}^{3/2}f\right\|_{L^2(X)}^2.
\end{aligned}$$

We include the proof because we need higher regularity than what is available via the standard theory of Lions and Magenes [21, chapter 3, section 8]. They prove

$$u(t) \in C([0,\infty);H) \quad \text{and} \quad u_t(t) \in C([0,\infty);L^2(X)),$$

but we need

$$u(t) \in C([0,\infty);V) \quad \text{and} \quad u_t(t) \in C([0,\infty);H).$$

Obtaining better regularity solely from [21] would essentially require $\langle A(t)f, g\rangle_H = \langle f, A(t)g\rangle_H$, which is not generally the case. To recover this identity, we introduce the time-indexed Hilbert spaces $H(t)$ with common domain $D(\mathcal{E})$ and equivlent norms

$$\|f\|_{H(t)}^2 := \|f\|_{L^2(X)}^2 + \mathcal{E}_t(f,f).$$

Consequently, $\langle A(t)f, g\rangle_{H(t)} = \langle f, A(t)g\rangle_{H(t)}$. Also, the Galerkin method we employ avoids the need to mimic the technical proofs of regularity in [21], which would be further complicated by the presence of $H(t)$.

Improving the regularity given by [21] will most likely require several strong assumptions, and checking that these assumptions are satisfied may not be feasible in practice. For instance, suppose for the moment that the operator $A(t)$ has the same set of eigenfunctions for all $t \in \mathbb{R}$, and also suppose that those eigenfunctions form a complete orthonormal set for $D(\mathbb{A})$. Then we can improve the regularity of the solution u given in [21] via the standard technique of revisiting and strengthening the Galerkin approximation procedure that generated u.

A.1 Additional conditions

We assume $D(\mathbb{A}) = D(A(t))$ for all $t \in \mathbb{R}$ and $D(\mathbb{A}^{3/2}) \subseteq D(A(0)^{3/2})$, and assume each containment in $W \subseteq V \subseteq H \subseteq L^2(X)$ is dense. We introduce the forms

$$a(t; f, g) := \langle A(t)f, A(t)g\rangle_{L^2(X)} + \mathcal{E}_t(f, g) \quad \text{and} \quad b(t; f, g) := \mathcal{E}_t(f, g) + \langle f, g\rangle_{L^2(X)}.$$

These forms are used to generate approximate solutions to (3.1). We assume the maps $t \mapsto a(t; \cdot, \cdot)$ and $t \mapsto b(t; \cdot, \cdot)$ are twice and three times continuously differentiable on $\mathbb{R}$, respectively. Similarly to Lions and Magenes [21], we assume

$$\left|\partial_t^k \langle A(t)f, A(t)g\rangle_{L^2(X)}\right| \leq C_k(T) \left\|f\right\|_V \left\|g\right\|_V , \tag{A1}$$

$$\alpha(T) \left\|f\right\|_V^2 \leq \lambda(T) \left\|f\right\|_H^2 + \left\|A(t)f\right\|_{L^2(X)}^2 , \tag{A2}$$

for $t \in [0, T]$ with $f, g \in V$ and $k = 0, 1, 2$, where $T > 0$ may be chosen arbitrarily and $\alpha(T) > 0$, $C_k(T), \lambda(T) \geq 0$. We also assume

$$A(t)f \in C(\mathbb{R}; L^2(X)) \tag{A3}$$

for $f \in V$, allowing us to show $u_{tt}(t)$ is continuous in $L^2(X)$.

A.2 Statement of the main proposition

Define the Hilbert spaces $\mathcal{V} = L^2([0,T];V)$, $\mathcal{H} = L^2([0,T];H)$ and $\mathcal{V}' = L^2([0,T];L^2(X))$ with inner products $(\cdot,\cdot)_{\mathcal{V}}$, $(\cdot,\cdot)_{\mathcal{H}}$ and $(\cdot,\cdot)_{\mathcal{V}'}$, respectively.

Definition A.1. *We say that $f(t) \in \mathcal{V}$ has a weak derivative $g(t) \in \mathcal{V}$ and data $f(0)$ if*

$$\left(f, \phi'\right)_{\mathcal{V}} + \left(g, \phi\right)_{\mathcal{V}} = -\left\langle f(0), \phi(0)\right\rangle_V$$

for every $\phi(t) \in C_c^\infty((-T,T);V)$. Similarly, $f(t) \in \mathcal{H}$ has a weak derivative $g(t) \in \mathcal{H}$ and data $f(0)$ if

$$\left(f, \phi'\right)_{\mathcal{H}} + \left(g, \phi\right)_{\mathcal{H}} = -\left\langle f(0), \phi(0)\right\rangle_H$$

for every $\phi(t) \in C_c^\infty((-T,T);H)$.

Since $C_c^\infty((0,T);V)$ is dense in $\mathcal{V}$, we see that weak derivatives in $\mathcal{V}$ are unique up to the data and sets of measure zero in t. Similarly, weak derivatives in $\mathcal{H}$ are unique up to the data and sets of measure zero.

For convenience, we restate Proposition 3.4 as

Proposition A.2. *Let (D1) - (D3) and (A1) - (A3) be satisfied, and let $T > 0$ be arbitrary. There exists a unique solution to (3.1) such that:*

$$0 = \|A(t)u(t) + u_t(t) + u_{tt}(t)\|_{L^2(X)} \;\; \textit{for} \;\; t \in [0,T], \tag{i}$$

$$u_t \textit{ is the weak derivative of } u \textit{ in } \mathcal{V} \textit{ with data } u_0, \tag{ii}$$

$$u_{tt} \textit{ is the weak derivative of } u_t \textit{ in } \mathcal{H} \textit{ with data } u_1, \tag{iii}$$

$$u(t) = u_0 + \int_0^t u_s(s)ds \in C([0,T];V), \tag{iv}$$

$$u_t(t) = u_1 + \int_0^t u_{ss}(s)ds \in C([0,T];H), \tag{v}$$

$$u_{tt}(t) \in C([0,T];L^2(X)). \tag{vi}$$

The remainder of this appendix is separated into subsections each handling different aspects of the proof of Proposition A.2.

A.3 Approximate solutions to (3.1):

We begin with the Faedo-Galerkin method; cf. Lions and Magenes [21, page 267]. Recall that u_0, u_1 are the data associated with (3.1). Let $W' \subseteq W$ be a complete orthonormal set for W. Since W is dense in V, there exists a countable subseteq $\{w_i\}_{i\in\mathbb{N}}$ of W' such that u_0 is in the closure of the linear span of $\{w_i\}_{i\in\mathbb{N}}$ with respect to W and u_1 is in the closure of the linear span of $\{w_i\}_{i\in\mathbb{N}}$ with respect to V. Thus we assume, without loss of generality, that $\{w_i\}_{i\in\mathbb{N}} = W'$.

Note that $w_1, \dots, w_m$ are linearly independent in H for all m. By assumption (D1), the norm $\|f\|^2_{H(t)} = b(t; f, f)$ is a norm on H that is equivalent to the original norm on H. Hence, $w_1, \dots, w_m$ are linearly independent in $H(t)$ norm for all m. For $t \in \mathbb{R}$, we apply Gram-Schmidt to $\{w_1, \dots, w_m\}$, with respect to the $H(t)$ norm, obtaining:

$$h_1(t) := w_1, \quad h_2(t) := w_2 - \frac{b(t; h_1, w_2)}{b(t; h_1, h_1)} h_1, \quad h_3(t) := w_3 - \frac{b(t; h_1, w_3)}{b(t; h_1, h_1)} h_1 - \frac{b(t; h_2, w_3)}{b(t; h_2, h_2)} h_2, \quad \dots$$

Let $v_i(t) := h_i(t)/\sqrt{b(t; h_i(t), h_i(t))}$. Observe that $h_1(t), \dots, h_m(t)$ are linearly independent in $H(t)$ for $t \in \mathbb{R}$, and $\mathcal{E}_t(f, g)$ is continuous for $t \in \mathbb{R}$, meaning that $b(t; h_i(t), h_i(t))$ remains bounded away from 0 for $t \in [-T, T]$, where $T \geq 0$ may be arbitrarily large. Thus, $\frac{1}{\|h_i(t)\|_{H(t)}} = \frac{1}{\sqrt{b(t;h_i(t),h_i(t))}} \in C^3(\mathbb{R})$ since $b(t, \cdot, \cdot) \in C^3(\mathbb{R})$ by assumption, giving

$$v_i(t) = \sum_{j=1}^{m} k_{ij}(t) w_j, \tag{A.1}$$

where each $k_{ij}(t) \in C^3(\mathbb{R})$. For each $m \in \mathbb{N}$, we define an approximate solution to (3.1)

$$u_m(t) := \sum_{i=1}^{m} G_{im}(t) v_i(t) = \sum_{i=1}^{m} g_{im}(t) w_i, \tag{A.2}$$

with $G_{im}(t)$ and $g_{im}(t) \in \mathbb{R}$. Since the $v_i(t)$ are orthonormal in $H(t)$, the system

$$\begin{cases} a(t; u_m(t), v_j(t)) + b(t; \partial_t u_m(t), v_j(t)) + b(t; \partial_t^2 u_m(t), v_j(t)) = 0, & t \in [0, T], \quad 1 \leq j \leq m, \\ (G_{im}, \partial_t G_{im})(0) = (\alpha_{im}, \beta_{im}), & 1 \leq i \leq m \end{cases}$$

is equivalent to

$$\begin{cases} M_1(t)G_m(t) + M_2(t)\partial_t G_m(t) + \partial_t^2 G_m(t) = 0, \quad t \in [0,T], \\ (G_m, \partial_t G_m)(0) = (\alpha_m, \beta_m), \end{cases} \tag{A.3}$$

where $G_m(t)$ is the column vector $(G_{im}(t))_{1\le i\le m}$, and the real-valued $m \times m$ matrices $M_1(t)$ and $M_2(t)$ are entry-wise once continuously differentiable for $t \in \mathbb{R}$ since $b(t,\cdot,\cdot) \in C^3(\mathbb{R})$ and $a(t,\cdot,\cdot) \in C^2(\mathbb{R})$ by assumption. Observe that (A.3) has a unique solution and $G_m(t) \in C^3([0,T];\mathbb{R}^m)$, so $G_{im}(t) \in C^3([0,T])$. Hence $g_{im}(t) \in C^3([0,T])$ by (A.1), and since $\{w_1,\ldots,w_m\}$ and $\{v_1(t),\ldots,v_m(t)\}$ have the same linear span, u_m is the unique solution to

$$\begin{cases} a(t; u_m(t), w_j) + b(t; \partial_t u_m(t), w_j) + b(t; \partial_t^2 u_m(t), w_j) = 0, & t \in [0,T], \quad 1 \le j \le m, \\ (g_{im}, \partial_t g_{im})(0) = (\nu_{im}, \eta_{im}), & 1 \le i \le m. \end{cases} \tag{A.4}$$

We choose data $\{(\nu_{im}, \eta_{im})\}_{1\le i\le m}$ such that

$$\sum_{i=1}^{m} \nu_{im} w_i \to u_0 \text{ in } W \text{ as } m \to \infty, \quad \text{and} \quad \sum_{i=1}^{m} \eta_{im} w_i \to u_1 \text{ in } V \text{ as } m \to \infty. \tag{A.5}$$

If u_0 or $u_1 = 0$, then we respectively choose $\nu_{im} = 0$ or $\eta_{im} = 0$ for all i and m.

A.4 Estimates for the strong energies of the approximate solutions:

Define the strong energy as $e(t; f) := a(t; f, f) + b(t; \partial_t f, \partial_t f)$. In this part, we derive estimates for $e(t; u_m)$ and $e(t; \partial_t u_m)$, subsequently deriving the estimate $\|u_m\|_V^2 + \|\partial_t u_m\|_V^2 + \|\partial_t^2 u_m\|_H^2 \le C(T)\left(\|u_0\|_W^2 + \|u_1\|_V^2\right)$ for $t \in [0,T]$ and $m \in \mathbb{N}$, where $C(T)$ is independent of m. The latter estimate will allow us to obtain a solution to (3.1).

Define $Y_1(t) := e(t; u_m) + \|u_m\|_H^2$, and proceed as in Lions and Magenes [21, page 267], substituting $2\partial_t u_m(t)$ in for w_j in (A.4), which is allowed by (A.2), to obtain

$$Y_1'(t) = (\partial_t e)(t; u_m) + 2\langle u_m, \partial_t u_m\rangle_H - 2b(t; \partial_t u_m, \partial_t u_m) \le (\partial_t e)(t; u_m) + \|u_m\|_H^2 + \|\partial_t u_m\|_H^2.$$

By assumptions (D3), (A1) and (A2), we have $(\partial_t e)(t; u_m) \le C_1(T)Y_1(t)$. Also, $\|\partial_t u_m\|_H^2 \le C_2\, b(t; \partial_t u_m, \partial_t u_m)$ by (D1). Hence, we obtain

$$Y_1'(t) \le C_3(T)Y_1(t),$$

meaning we have the first estimate for the strong energy,

$$e(t; u_m) + \|u_m\|_H^2 \le C_4(T) \left(e(0; u_m(0)) + \|u_m(0)\|_H^2\right). \tag{A.6}$$

Therefore, we have

$$\|u_m\|_V^2 + \|\partial_t u_m\|_H^2 \le C_5(T) \left(\|u_m(0)\|_V^2 + \|\partial_t u_m(0)\|_H^2\right) \le C_6(T) \left(\|u_0\|_V^2 + \|u_1\|_H^2\right) \tag{A.7}$$

by (A.5) and assumptions (A1), (A2) and (D1).

Now we derive the second estimate for the strong energy. Define $Y_2(t) := e(t; \partial_t u_m) + \|\partial_t u_m\|_H^2$. Apply ∂_t to (A.4) and then substitute $2\partial_t^2 u_m(t)$ in for w_j to get

$$Y_2'(t) + P_1'(t) = 3(\partial_t e)(t; \partial_t u_m) + 2 \left\langle \partial_t u_m, \partial_t^2 u_m \right\rangle_H - 2b(t; \partial_t^2 u_m, \partial_t^2 u_m) + P_2(t),$$

where $P_1(t) := 2(\partial_t a)(t; u_m, \partial_t u_m) + (\partial_t b)(t; \partial_t u_m, \partial_t u_m)$, and $P_2(t) := 2(\partial_t^2 a)(t; u_m, \partial_t u_m) + (\partial_t^2 b)(t; \partial_t u_m, \partial_t u_m) - 4(\partial_t b)(t; \partial_t^2 u_m, \partial_t^2 u_m)$. Similarly to above,

$$Y_2'(t) + P_1'(t) \le C_7(T)Y_2(t) + P_2(t) \le C_8(T) \left(Y_2(t) + Y_1(t)\right) \tag{A.8}$$

since $P_2(t) \le C_9(T) \left(Y_2(t) + Y_1(t)\right)$ by reasoning similar to above. Next, observe that $4|(\partial_t a)(t; u_m, \partial_t u_m)| \le a(t; \partial_t u_m) + \|\partial_t u_m\|_H^2 + C_{10}(T) \left(a(t; u_m) + \|u_m\|_H^2\right)$ by (A1) and (A2), giving $2|P_1(t)| \le Y_2(t) + C_{11}(T)\, Y_1(t)$. Hence we see

$$2P_1(t) - 2P_1(0) \ge -Y_2(t) - Y_2(0) - C_{11}(T) \left(Y_1(t) + Y_1(0)\right) \tag{A.9}$$

Change the variable in (A.8) from t to s and integrate is s on $[0,t]$, with $t \leq T$, and then apply (A.9) to obtain

$$\frac{1}{2}Y_2(t) - \frac{3}{2}Y_2(0) \leq C_8(T)\int_0^t Y_2(s) + Y_1(s)ds + \frac{C_{11}(T)}{2}\left(Y_1(t) + Y_1(0)\right).$$

Bound Y_1 from above by $C_6(T)\left(\|u_0\|_V^2 + \|u_1\|_H^2\right)$ via (A.7), and apply Gronwall's lemma to obtain the second estimate for the strong energy

$$e(t;\partial_t u_m) + \|\partial_t u_m\|_H^2 \leq C_{12}(T)\left(\|u_0\|_V^2 + \|u_1\|_H^2 + e(0;\partial_t u_m(0)) + \|\partial_t u_m(0)\|_H^2\right). \tag{A.10}$$

Substitute $\partial_t^2 u_m(t)$ into (A.4) for w_j, let $t = 0$, and note $\langle A(0)f, A(0)g\rangle_{L^2(X)} = \langle A(0)^{3/2}f, A(0)^{1/2}g\rangle_{L^2(X)}$ for $f = u_m(0)$ and $g = \partial_t^2 u_m(0)$. Thus since $D(\mathbb{A}^{3/2}) \subseteq D(A(0)^{3/2})$, we see

$$\begin{aligned} b(0;\partial_t^2 u_m(0), \partial_t^2 u_m(0)) &= -b(0;\partial_t u_m(0), \partial_t^2 u_m(0)) - a(0; u_m(0), \partial_t^2 u_m(0)) \\ &\leq C\left(\|u_m(0)\|_W^2 + \|\partial_t u_m(0)\|_V^2\right) + \frac{1}{2}b(0;\partial_t^2 u_m(0), \partial_t^2 u_m(0)). \end{aligned}$$

Hence the second estimate for the strong energy (A.10) and the choice of data (A.5) give

$$\|\partial_t u_m\|_V^2 + \|\partial_t^2 u_m\|_H^2 \leq C_{13}(T)\left(\|u_0\|_W^2 + \|u_1\|_V^2\right). \tag{A.11}$$

A.5 Existence of solution to (3.1) and regularity:

Combining (A.7) and (A.11), we have $\|u_m\|_V^2 + \|\partial_t u_m\|_V^2 + \|\partial_t^2 u_m\|_H^2 \leq C(T)\left(\|u_0\|_W^2 + \|u_1\|_V^2\right)$ for $m \in \mathbb{N}$, where $C(T)$ is independent of m. By the reflexivity of $\mathcal{V}$ and $\mathcal{H}$, we can find a subsequence $\{u_{m_k}\}$ of $\{u_m\}$ such that

$$u_{m_k} \to u \text{ weakly in } \mathcal{V}, \quad \partial_t u_{m_k} \to \chi_1 \text{ weakly in } \mathcal{V}, \quad \partial_t^2 u_{m_k} \to \chi_2 \text{ weakly in } \mathcal{H}.$$

Hence we obtain Proposition A.2 (ii) with $u_t := \chi_1$ since $\partial_t u_m$ is the weak derivative of u_m in $\mathcal{V}$ with data $u_m(0)$. Since $\left(\cdot,\cdot\right)_{\mathcal{H}}$ is a bounded bilinear form on $\mathcal{V} \times \mathcal{V}$ and $\mathcal{V}$ is dense in

$\mathcal{H}$, we have $\partial_t u_{m_k} \to u_t$ weakly in $\mathcal{H}$. Thus we also have Proposition A.2 (iii) with $u_{tt} := \chi_2$ since $\partial_t^2 u_m$ is the weak derivative of $\partial_t u_m$ in $\mathcal{H}$ with data $\partial_t u_m(0)$.

We proceed to show parts (iv) and (v). Approximate u_t in $\mathcal{V}$ with functions in $C_c^\infty((0,T);V)$ to show that u_t is also the weak derivative of $U := u_0 + \int_0^t u_s(s)ds$ in $\mathcal{V}$ with data u_0. Thus we see $\left(u - U, \phi'\right)_{\mathcal{V}} = 0$ for every $\phi(t) \in C_c^\infty((-T,T);V)$. Let $\eta(t) \in C_c^\infty((0,T);V)$ and define $\Psi(t) := \int_{-T}^t \eta(s) - \eta(-s)ds \in C_c^\infty((-T,T);V)$. Then note $0 = \left(u - U, \Psi'\right)_{\mathcal{V}} = \left(u - U, \eta\right)_{\mathcal{V}}$, implying that $\|u - U\|_{\mathcal{V}} = 0$. Therefore, we alter u on a set of measure zero in t, demonstrating Proposition A.2 (iv). Part (v) similarly follows.

Now we will show parts (i) and (vi). Recall that we are assuming $\{w_i\}_{i\in\mathbb{N}}$ is a complete orthonormal set for $\mathcal{V}$. Consider the set S_T of all functions of the form $\psi_p(t) := \sum_{j=1}^p \phi_j(t)w_j$, where $\phi_j(t) \in C_c^\infty((0,T);\mathbb{R})$ for each j and p ranges over $\mathbb{N}$. Note that S_T is dense in $\mathcal{V}$. Now for $p \le m_k$,

$$0 = \int_0^T a(t; u_{m_k}, \psi_p) + b(t; \partial_t u_{m_k} + \partial_t^2 u_{m_k}, \psi_p)dt \tag{A.12}$$

by (A.4). Define the operator $\mathcal{A}_t : \mathcal{V}' \to \mathcal{V}'$, with domain $\mathcal{V}$, via $\left(\mathcal{A}_t f, g\right)^2_{\mathcal{V}'} := \int_0^T \langle A(t)f(t), g(t)\rangle_{L^2(X)}\, dt$. Note that $\int_0^T a(t;\cdot,\cdot)dt$ is a bounded bilinear form on $\mathcal{V}\times\mathcal{V}$, allowing us to replace u_{m_k} with u in (A.12) via taking $k \to \infty$. Then the density of S_T allows us to replace ψ_p with $v \in \mathcal{V}$. Therefore, we see

$$0 = \left(\mathcal{A}_t u + u_t + u_{tt}, (\mathcal{A}_t + 1)v\right)_{\mathcal{V}'},$$

and by a forthcoming lemma, the operator $\mathcal{A}_t + 1$ is surjective, hence $0 = \|\mathcal{A}_t u + u_t + u_{tt}\|_{\mathcal{V}'}$. We alter u_{tt} on a set of measure zero in t to obtain Proposition A.2 (i).

By the regularity of u, i.e., part (iv) and assumption (A1), $\|A(t)(u(t) - u(s))\|_{L^2(X)} \to 0$ as $t \to s$, and assumption (A3) gives $\|(A(t) - A(s))u(s)\|_{L^2(X)} \to 0$ as $t \to s$. Thus $A(t)u(t) \in C([0,T];L^2(X))$. Also, $u_t(t) \in C([0,T];L^2(X))$ by part (v). Therefore $u_{tt}(t) = -u_t(t) - A(t)u(t) \in C([0,T];L^2(X))$, proving part (vi).

We now give the promised lemma. Denote the range and null space of an operator T by $\mathcal{R}(T)$ and $\mathcal{N}(T)$, respectively.

Lemma A.3. *The operator $\mathcal{A}_t + 1 : \mathcal{V}' \to \mathcal{V}'$ is surjective and self-adjoint.*

Proof. Let $\{y_j\}_{j\in J}$ and $\{\phi_k(t)\}_{k\in K}$ be complete orthonormal sets for $L^2(X)$ and $L^2([0,T];\mathbb{R})$, respectively. Define $Y_{j,k}(t) := \phi_k(t)y_j$, and note that $\{Y_{j,k}\}_{(j,k)\in J\times K}$ is a complete orthonormal set for $\mathcal{V}'$.

Note that $A(t)+1$ is self-adjoint and injective, giving $L^2(X) = \overline{\mathcal{R}(A(t)+1)} \oplus \mathcal{N}((A(t)+1)^*) = \mathcal{R}(A(t)+1)$ since $\mathcal{R}(A(t)+1)$ is closed. Thus $(A(t)+1)^{-1} : L^2(X) \to V$ is bounded. Let $v(t) := (A(t)+1)^{-1}y_j$ and observe

$$\begin{aligned}\|v(t)-v(s)\|_V \leq & C(T)\left(\|v(t)-v(s)\|_H + \|A(t)\,(v(t)-v(s))\|_{L^2(X)}\right)\\ \leq & C_1(T)\|(A(t)+1)\,(v(t)-v(s))\|_{L^2(X)}\end{aligned}$$

via assumptions (A2), (D1), the nonnegativity of $A(t)$ and the identity $\mathcal{E}_t(f,g) = \langle A(t)f,g\rangle_{L^2(X)}$ for $f\in D(A(t))$. Also, $(A(t)+1)\,(v(t)-v(s)) = y_j - (A(t)+1)(A(s)+1)^{-1}y_j \to 0$ in $L^2(X)$ as $t\to s$ by assumption (A3). Therefore, $v(t)$ is continuous in V, so $v(t)\in\mathcal{V}$, giving y_j and hence $Y_{j,k}\in\mathcal{R}(\mathcal{A}_t+1)$. Since $A(t)+1$ is closed, we have $\mathcal{A}_t+1$ is closed. Hence $\mathcal{R}(\mathcal{A}_t+1)$ is closed, meaning that $\mathcal{R}(\mathcal{A}_t+1) = \mathcal{V}'$ since $\{Y_{j,k}\}_{(j,k)\in J\times K} \subseteq \mathcal{R}(\mathcal{A}_t+1)$. Thus $\mathcal{A}_t+1$ is surjective.

Consequently, $\mathcal{N}((\mathcal{A}_t+1)^*) = \{0\}$. To show $\mathcal{A}_t+1$ is self-adjoint, we only need to show $D((\mathcal{A}_t+1)^*) \subseteq D(\mathcal{A}_t+1)$ since $\mathcal{A}_t+1$ is symmetric. Let $x\in D((\mathcal{A}_t+1)^*)$, and note that $(\mathcal{A}_t+1)^*x = (\mathcal{A}_t+1)x_1 = (\mathcal{A}_t+1)^*x_1$ for some $x_1\in\mathcal{V}$ because $\mathcal{A}_t+1$ is surjective. Therefore, $x_1 = x\in D(\mathcal{A}_t+1)$ since $x-x_1\in\mathcal{N}((\mathcal{A}_t+1)^*)$. ∎

A.6 The energy inequalities and uniqueness for the solution to (3.1):

We now prove standard energy inequalities, and as a consequence, we prove that the solution to (3.1) is unique, completing Proposition A.2. As in section 3.2, we have the following notation for the energy and total energy:

$$E(t;f) = \frac{1}{2}\int_X |\partial_t f(x,t)|^2 dm(x) + \frac{1}{2}\mathcal{E}_t(f,f), \quad T_E(t;f) = E(t;f) + \frac{1}{2}\int_X |f(x,t)|^2 dm(x).$$

Take the $L^2(X)$ inner product of (3.1) with $2u_t + u$ to obtain

$$\partial_t \left(E(t;u) + T_E(t;u) + \langle u, u_t \rangle_{L^2(X)} \right) + 2E(t;u) - (\partial_t \mathcal{E}_t)(u,u) = 0.$$

Then assumption (D3) gives $(\partial_t \mathcal{E}_t)(u,u) \leq 2c_3 E(t;u)$. Hence we have the first energy inequality

$$T_E(t;u) \leq C(c_3, T)\, T_E(0;u) \tag{A.13}$$

for $t \in [0,T]$. As an immediate consequence, this completes Proposition A.2.

To prove the second energy inequality, take the $L^2(X)$ inner product of (3.30) with the difference quotient u_t^h to obtain

$$\partial_t \left(E(t;u^h) \right) + \|\partial_t u^h\|^2_{L^2(X)} - \frac{1}{2}(\partial_t \mathcal{E}_t)(u^h, u^h) + (D^h \mathcal{E}_t)(w, \partial_t u^h) = 0.$$

Similarly to above, $(\partial_t \mathcal{E}_t)(u^h, u^h) \leq 2c_3 E(t;u^h)$, and $\|\partial_t u^h\|_{L^2(X)} \geq 0$. Hence,

$$\partial_t \left(e^{-2c_3 t} E(t;u^h) \right) \leq -\partial_t \left(e^{-2c_3 t} (D^h \mathcal{E}_t)(w, u^h) \right) + e^{-2c_3 t} Q_h(t), \tag{A.14}$$

where $Q_h(t) = -2c_3 (D^h \mathcal{E}_t)(w,u^h) + (D^h(\partial_t \mathcal{E}_t))(w,u^h) + (D^h \mathcal{E}_t)(\partial_t w, u^h)$. Integrate both sides of (A.14) in t from 0 to r, where $r \leq T$, and then take $h \to 0$. We apply dominated convergence, which is permitted via the regularity of u, and we see

$$\left(e^{2c_3(r-t)} E(t;\partial_t u) \right)\Big|_{t=0}^{r} \leq -\left(e^{2c_3(r-t)} (\partial_t \mathcal{E}_t)(u, \partial_t u) \right)\Big|_{t=0}^{r} + \int_0^r e^{2c_3(r-t)} Q(t)\,dt, \tag{A.15}$$

where $Q(t) = -2c_3 (\partial_t \mathcal{E}_t)(u, \partial_t u) + (\partial_t^2 \mathcal{E}_t)(u, \partial_t u) + (\partial_t \mathcal{E}_t)(\partial_t u, \partial_t u)$. Estimate

$$|(\partial_t \mathcal{E}_t)(u, \partial_t u)| \leq \frac{1}{4} \mathcal{E}_t(\partial_t u, \partial_t u) + 4(c_3)^2 \mathcal{E}_t(u,u) \leq \frac{1}{2} E(t;\partial_t u) + C_1(c_3,T)\, T_E(0;u)$$

via assumption (D3) and the Cauchy-Schwarz inequality (C-S), followed by (A.13). Similarly, estimate

$$Q(t) \leq C(c_3, c_4, T) \left(E(t;\partial_t u) + T_E(0;u) \right).$$

via assumption (D3). Therefore, inequality (A.15) gives

$$\frac{1}{2}E(r;\partial_r u) \leq C_1(c_3,c_4,T)\left(E(0;\partial_r u) + T_E(0;u) + \int_0^r E(t;\partial_t u)dt\right).$$

Apply Gronwall's lemma to obtain the second energy inequality

$$E(t;\partial_t u) \leq C_2(c_3,c_4,T)\left(E(0;\partial_t u) + T_E(0;u)\right) \tag{A.16}$$

for $t \in [0,T]$.

B Proofs of lemmas 3.8 and 3.9

Let $\mathcal{E}$ be the reference Dirichlet form discussed in subsection 3.1.1, and let $\mathcal{E}_t$ be the time-dependent Dirichlet form corresponding to the operator $A(t)$ from (3.1). Recall the space $H = D(\mathcal{E})$ has norm $(\mathcal{E}(f,f)+\|f\|^2_{L^2(X)})^{1/2}$, and $H = D(\mathcal{E}_t)$ for all $t \in \mathbb{R}$. Also, $\mathcal{E}$ is a regular form, meaning that for any $f \in H$, we can find $f_n \in H \cap C_c(X)$ such that $\|f_n - f\|_H \to 0$ as $n \to \infty$.

Proof of lemma 3.8. We prove the lemma for $i = 1$ since the proof for $i = 2$ is similar. Let $g, h \in H$ and $f \in H \cap L^2(X, d\Gamma_t(g,g))$. Since $\mathcal{E}$ is a regular form, we can find $g_n, h_n \in H \cap C_c(X)$ such that $\|g_n - g\|_H$ and $\|h_n - h\|_H \to 0$ as $n \to \infty$. Define $f_m := \min\{\max\{f, -m\}, m\}$ for $m \in \mathbb{N}$, and note that $f_m \in H$ via [6, Theorem 1.4.2 (iii)]. From subsection 3.1.1, recall the identity $d(\partial_t\Gamma_t)(g_n, h_n) = \sum_{j=1}^J \alpha_{1,j} d\Gamma_t^j(g_n, h_n)$. Define $\Gamma_t^0 := \Gamma_t$, where Γ_t is the energy measure form associated with $\mathcal{E}_t$. For $0 \le j \le J$, define the functions

$$F^j_{m,n}(t) := \int_X f_m d\Gamma^j_t(g_n, h_n), \quad F^j_m(t) := \int_X f_m d\Gamma^j_t(g, h), \quad F^j(t) := \int_X f d\Gamma^j_t(g, h).$$

Weak integration-by-parts (3.9) and the fundamental theorem of calculus give

$$F^0_{m,n}(t) - F^0_{m,n}(0) = \int_0^t (\partial_s\mathcal{E}_s)(f_m g_n, h_n) + (\partial_s\mathcal{E}_s)(f_m h_n, g_n) - (\partial_s\mathcal{E}_s)(g_n h_n, f_m) ds. \tag{B.1}$$

Thus $F^0_{m,n}(t) \in C^1(\mathbb{R})$ since we assume $(\partial_s\mathcal{E}_s) \in C(\mathbb{R})$. By assumption (D2) and weak integration-by-parts (3.9), the integrand of the RHS of (B.1) is equal to

$$\sum\nolimits_{j=1}^J \alpha_{1,j} \left(\mathcal{E}^j_s(f_m g_n, h_n) + \mathcal{E}^j_s(f_m h_n, g_n) - \mathcal{E}^j_s(g_n h_n, f_m)\right) = \sum\nolimits_{j=1}^J \alpha_{1,j} F^j_{m,n}(s),$$

which is continuous since, as above, $(\partial_s\mathcal{E}_s) \in C(\mathbb{R})$. Hence,

$$F^0_{m,n}(t) - F^0_{m,n}(0) = \int_0^t \sum\nolimits_{j=1}^J \alpha_{1,j} F^j_{m,n}(s) ds. \tag{B.2}$$

Therefore, the lemma will be proved once we show $F^j_{m,n}(t) \to F^j(t)$ uniformly as $m,n\to\infty$, where $0 \le j \le J$, since this will mean that $\sum_{j=1}^{J} \alpha_{1,j} F^j(s) = \int_X f d\left(\partial_s \Gamma_s\right)(g,h)$ is a uniform limit of continuous functions.

Recall that $d\Gamma_t(g,g) \le c_2 d\Gamma(g,g)$ by assumption (D1). Hence the Cauchy-Schwarz inequality (C-S) and assumption (D3) give

$$\left|F^j_m(t) - F^j(t)\right|^2 \le \int_X |f_m - f|^2 d\Gamma^j_t(g,g) \int_X d\Gamma^j_t(h,h) \le C \int_X |f_m - f|^2 d\Gamma(g,g) \int_X d\Gamma(h,h).$$

Thus since $f \in L^2(X, d\Gamma(g,g))$, we have $F^j_m(t) \to F^j(t)$ uniformly as $m \to \infty$ via dominated convergence. Similarly, $\left|F^j_{m,n}(t) - F^j_m(t)\right|^2 \le C(m)\left(\|g_n - g\|^2_H \|h_n\|^2_H + \|g_n\|^2_H \|h_n - h\|^2_H\right)$, meaning $F^j_{m,n}(t) \to F^j_m(t)$ uniformly as $n \to \infty$. □

Proof of lemma 3.9. Since $\mathcal{E}$ is a regular form, we can find $g_n(x), h_m(x) \in H \cap C_c(X)$ such that $\|g_n - g\|_H$ and $\|h_m - h\|_H \to 0$ as $m,n \to \infty$. Weak integration-by-parts (3.9) gives

$$\int_X f d\Gamma_t(g_n, h_m) = \mathcal{E}_t\left(f g_n, h_m\right) + \mathcal{E}_t\left(f h_m, g_n\right) - \mathcal{E}_t\left(g_n h_m, f\right)$$
$$\int_X g_n d\Gamma_t(f, h_m) = \mathcal{E}_t\left(f g_n, h_m\right) + \mathcal{E}_t\left(g_n h_m, f\right) - \mathcal{E}_t\left(f h_m, g_n\right).$$

Combining the left-hand sides of these identities gives

$$\int_X f d\Gamma_t(g_n, h_m) + \int_X g_n d\Gamma_t(f, h_m) = 2\mathcal{E}_t(f g_n, h_m), \tag{B.3}$$

and note $|\mathcal{E}_t(f g_n, h_m)|^2 \le (c_2)^2 \mathcal{E}(f g_n, f g_n)\mathcal{E}(h_n, h_m)$ by assumption (D1). Take $m \to \infty$, and notice $\mathcal{E}_t\left(f g_n, h\right) = \int_X f g_n (A(t)h) dm(x)$. Take $n \to \infty$. □

Vita

Montgomery Taylor was born in New Brunswick, New Jersey. He developed a love for mathematics at age thirteen, when he began to derive combinatorial identities from scratch. Beginning at age seventeen, Montgomery worked as an electrician for two years. Afterwords, he earned his A.S. in Engineering from Walter's State Community College in 2005. He then transferred to the University of Tennessee, Knoxville where he discovered a fondness for real analysis and abstract algebra. In the summer of 2007, he attended a research experience for undergraduates at Wabash College, where he grappled with problems involving zero-divisor graphs. In 2009, he earned his B.S. in Mathematics and the *John H. Barrett Prize for Excellence in Mathematics* from the University of Tennessee. After Montgomery earned his M.A. in Mathematics from the University of California, San Diego in 2011, he and Hutch Brock began a professional math tutoring service. Montgomery met his future wife Kat Ashdown and began attending graduate school at the University of Tennessee in 2013. He earned the *Graduate Student Academic Achievement Award* and published his first paper in 2018.